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An Understanding of the Physical Solutions and the Blow-Up Phenomenon for Nonlinear Noisy Leaky Integrate and Fire Neuronal Models

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Abstract. The Nonlinear Noisy Leaky Integrate and Fire neuronal models are mathematical models that describe the activity of neural networks. These models have been studied at a microscopic level, using Stochastic Differential Equations, and at a mesoscopic/macroscopic level, through the mean field limits using Fokker-Planck type equations. The aim of this paper is to improve their understanding, using a numerical study of their particle systems. This allows us to go beyond the mesoscopic/macroscopic description. We answer one of the most important open questions about these models: what happens after all the neurons in the network fire at the same time? We find that the neural network converges towards its unique steady state, if the system is weakly connected. Otherwise, its behaviour is more complex, tending towards a stationary state or a "plateau" distribution (membrane potentials are uniformly distributed between reset and threshold values). To our knowledge, these distributions have not been described before for these nonlinear models. In addition, we analyse in depth the behaviour of the classical and physical solutions of the Stochastic Differential Equations and, we compare it with what is already known about the classical solutions of Fokker-Planck equation. In this way, our numerical analysis, based on the microscopic scale, allows us to explain not only what happens after the explosion phenomenon, but also, how the physical solutions of the Fokker-Planck equation are. This notion of solution, for the Fokker-Planck equation, has not been studied to date.

AMS subject classifications: 35K60, 35Q92, 82C31, 82C32, 92B20

Key words: Neuronal networks, Leaky Integrate and Fire models, Fokker-Planck equation, stochastic differential equation, blow-up phenomena, plateau distribution.

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1 Introduction

Nonlinear Noisy Leaky Integrate and Fire neuronal (NNLIF) models are one of the simplest models used to describe the behaviour of a neuronal network [2–4, 23, 28, 37]. In recent years, NNLIF models have been studied from a mathematical point of view; at the microscopic level, using Stochastic Differential Equations (SDE) [16, 17, 24], and at a mesoscopic/macroscopic level, through the mean field limits using Fokker-Planck type equations (FPE) [5, 7–10, 12, 22, 36]. The considerable amount of publications and unanswered questions on these models reveal their high mathematical complexity, despite their simplicity.

The aim of this paper is to advance the understanding of the NNLIF models. We analyse in depth the behaviour of the classical and physical solutions of the Stochastic Differential Equations and we compare it with what is already known about the Fokker-Planck equation, using a numerical study of their particle systems. This allows us to understand what happens in the neural network when an explosion occurs in finite time, which is one of the most important open problems about this kind of models.

1.1 Stochastic differential equation and Fokker-Planck equation

Let us consider a large set of $\mathcal{N}$ identical neurons which are connected to each other in a network and described by the Nonlinear Noisy Leaky Integrate and Fire (NNLIF) model. This model represents the network activity in relation to the *membrane potential*, which is the potential difference on both sides of the neuronal membrane. The membrane potential $V_i(t)$ of a single neuron i is given by [3, 4, 15, 35]:

$$C_m \dot{V}_i(t) = -g_L(V_i(t) - V_L) + I_i(t), \quad i = 1, \cdots, \mathcal{N}, \tag{1.1}$$

where C_m is the capacitance of the membrane, g_L is the leak conductance, V_L is the resting potential and $I_i(t)$ are the synaptic currents. These currents are produced by the local and external synapses, i.e. they are the sum of spikes received from C neurons (inside and outside the neuron network):

$$I_i(t) = \sum_j \sum_k J_{ij} \delta(t - t_{ik}^j - d). \tag{1.2}$$

The Dirac delta $\delta(t - t_{ik}^j - d)$ models the input contribution of each spike; t_{ik}^j is the time when the k-th spike of the j-th neuron took place, J_{ij} is the synaptic strength (positive value for excitatory neurons and negative value for inhibitory ones), and d in the argument is the synaptic delay.

A neuron spikes when its membrane voltage reaches the firing threshold value V_F. Then, the neuron discharges itself by sending a spike perturbation over the network, and its membrane potential is set to the reset value V_R. The relation between the three values V_L, V_F and V_R is the following: $V_L < V_R < V_F$.