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The Method of Fundamental Solution for a Radially Symmetric Heat Conduction Problem with Variable Coefficient

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Abstract. We consider an inverse heat conduction problem with variable coefficient on an annulus domain. In many practice applications, we cannot know the initial temperature during heat process, therefore we consider a non-characteristic Cauchy problem for the heat equation. The method of fundamental solutions is applied to solve this problem. Due to ill-posedness of this problem, we first discretize the problem and then regularize it in the form of discrete equation. Numerical tests are conducted for showing the effectiveness of the proposed method.

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1 Introduction

In the process of heat conduction on an annulus domain, the following heat equation with variable coefficient is a good mathematical model in many situations:

$$\frac{\partial u}{\partial t}(r,t)=a(t)\left(\frac{\partial^2 u}{\partial r^2}(r,t)+\frac{1}{r}\frac{\partial u}{\partial r}(r,t)\right),\quad r_0<r<R,\quad 0<t<T. \tag{1.1}$$

With suitable initial and boundary conditions, the heat conduction can be specified. For example, consider the following initial boundary value problem:

$$\frac{\partial u}{\partial t}(r,t)=a(t)\left(\frac{\partial^2 u}{\partial r^2}(r,t)+\frac{1}{r}\frac{\partial u}{\partial r}(r,t)\right),\quad r_0<r<R,\quad 0<t<T;$$

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$$u(r,0)=u_0(r);$$
$$u(r_0,t)=u(R,t)=0.$$

If the suitable regularity of the data $u_0(r)$ and the coefficient $a(t)$ is given, one can prove the well-poseness of the classical problem for the solution u. Generally, we call it the direct problem. But in many engineering problems, the initial condition cannot be obtained because one cannot know when the heat conduction begins. Usually we want to know the temperature on inaccessible interior boundary of a pipe, but only the measurements from the outer boundary are available, this can be formulated as an inverse problems with $a(t)>0$:

$$\frac{\partial u}{\partial t}(r,t)=a(t)\left(\frac{\partial^2 u}{\partial r^2}(r,t)+\frac{1}{r}\frac{\partial u}{\partial r}(r,t)\right), \quad r_0<r<R, \quad 0<t<T \tag{1.2}$$

with Dirichlet and Neumann data on the known boundary R (accessible for data measurement)

$$u(R,t)=g(t), \qquad 0<t\leq T, \tag{1.3}$$
$$\frac{\partial u}{\partial r}(R,t)=h(t), \qquad 0<t\leq T. \tag{1.4}$$

In this paper, we want to recover the initial temperature $u(r,0)$ and the inner boundary temperature $u(r_0=0,t)$. This inverse heat conduction problem is called non-characteristic Cauchy problem mathematically. It is a well-known severely ill-posed problems. For ill-posed problems of mathematics physical equations, there are two ways to deal with. the first way is to discretize the problem first and then regularize it. The other way is to regularize it first and then discretize it. For example, for the fist way we can use the finite difference method to discretize the PDE-model problem and then an ill-posed discritized system is obtained and one can use all kinds of regularization methods [1,2] (e.g. truncated singular value decomposition) to treat it. For the second way, we make the ill-posed PDE-model problem to be well-posed, and then one can use all kinds of numerical methods for solving the well-posed problem. In this paper, we use the first way. To do that, we use the MFS to discretize the problem, and then we use the Tikhonov method to regularize it. As in [3], we emphasize the advantages of the MFS:"it is simple and feasible to handle various boundary conditions; it is easier to solve inverse problems than boundary element method, finite element method and finite difference method; it is applicable to solve more complex problems."

The MFS is a meshless method. It is first used to solve inverse heat conduction problem in [4] and then it is extended to solve all kinds of inverse problems [5-13]. It has been proved that the MFS with regularization methods is an efficient method for solving inverse and ill-posed problems.

A problem which is similar to (1.2)-(1.4) has been considered in [3]. But the problem in [3] is considered in the Cartesian coordinates. The fundamental solutions in this paper