

A Sparse Grid Discrete Ordinate Discontinuous Galerkin Method for the Radiative Transfer Equation

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Abstract. The radiative transfer equation is a fundamental equation in transport theory and applications, which is a 5-dimensional PDE in the stationary one-velocity case, leading to great difficulties in numerical simulation. To tackle this bottleneck, we first use the discrete ordinate technique to discretize the scattering term, an integral with respect to the angular variables, resulting in a semi-discrete hyperbolic system. Then, we make the spatial discretization by means of the discontinuous Galerkin (DG) method combined with the sparse grid method. The final linear system is solved by the block Gauss-Seidel iteration method. The computational complexity and error analysis are developed in detail, which show the new method is more efficient than the original discrete ordinate DG method. A series of numerical results are performed to validate the convergence behavior and effectiveness of the proposed method.

AMS subject classifications: 65N30, 65N35, 65N15, 65R20

Key words: Radiative transfer equation, sparse grid method, discrete ordinate method, discontinuous Galerkin method.

1 Introduction

Radiation transport is a physical process of energy transfer in the form of electromagnetic radiation which is affected by absorption, emission and scattering as it passes through the background materials. The radiative transfer equation (RTE) is an important mathematical model used to describe these interactions, finds applications in a wide variety of subjects, including neutron transport, heat transfer, optics, astrophysics, inertial confinement fusion, and high temperature flow systems, see for examples [2, 12, 16, 20, 27, 40, 47].

The RTE can be viewed as a hyperbolic-type integro-differential equation. Even for the stationary monochromatic RTE, it is five-dimensional in the phase space, and hence

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cannot have a closed-form solution in general. Thus, the numerical solution of the equation is unavoidable and critical in applications. In history, the Monte-Carlo method is a typical approach for numerical simulation (cf. [11] and the references therein). The advantage is its simplicity and dimension-free convergence, and the weakness is its heavy computational cost and slow convergence. Until now, there have developed many other numerical methods as well. For the angular discretization, the typical methods include discrete ordinate methods (or S_N methods) and spherical harmonic methods (or P_N method); for the spatial discretization, the typical methods include finite difference methods and finite element methods. We refer to [6, 7, 12, 18, 20, 27, 30, 31] and references therein for details. Due to the flexibility and easy implementation, the discrete ordinate method is frequently used for angular discretization in practice. If the spatial domain is regular, this semi-discrete method is also discretized by the Chebyshev spectral method in [5, 17, 28]. In the case of irregular domains, the meshless method is also used in [29, 34, 42]. In recent years, the positivity-preserving schemes are developed very technically in [15, 45, 48] and an adaptive moving mesh discontinuous Galerkin method is also reported in [47]. For numerical solvers such as source iteration and multigrid algorithms, one can refer to [1, 13, 36, 38].

On the other hand, except the Monte-Carlo method, all the methods mentioned above solve the problems with reduced dimensions. In this paper, we intend to attack the problem in its original form with 3-spatial variables and 2-angular variables. In this case, most usual methods suffer from the so-called “Curse of Dimensionality”, which indicates the low rate of convergence in terms of number of degrees of freedom due to the high dimensionality of the underlying problem. To the best of our knowledge, the sparse grid method, also called the sparse tensor product method, is an effective way to overcome the bottleneck. Historically, the idea of sparse grids can be traced back to Smoljak’s construction of multivariate quadrature formulas using combinations of tensor products of suitable one-dimensional formulas (cf. [19, 39]). More recently, the systematic and thorough studies on the method can be found in [19, 22, 23, 46]. In addition, several sparse grid methods are devised in [21, 44] for solving the RTE based on the conforming spatial discretization. However, according to the computational experience, it is preferable to use the discontinuous Galerkin (DG) method for spatial discretization for hyperbolic problems (cf. [9, 10, 14]), in order to capture non-smooth physical solutions. In [43], the sparse grid technique combined with the DG method has been developed for elliptic equations. This method is also applied to transport equations in [24, 25], but the scattering effect is not considered. The adaptive analogues of their methods are also given in [25, 41].

In this paper, we are going to propose and analyze a sparse grid DG method to solve the RTE, following the ideas in [27] and [24]. Unlike the studies in [21, 44], the DG method will be used to carry out the spatial discretization. And different from [24], we will discuss in detail the efficient solution of the 5-dimensional RTE with scattering effect. Concretely speaking, the discrete ordinate technique is first applied to discretize the scattering term, an integral with respect to the angular variables, by simply picking several directions spanning the solid angle, resulting in a semi-discrete coupled hyperbolic system.