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Geometric Estimates of the First Eigenvalue of (p,q)-elliptic Quasilinear System Under Integral Curvature Condition

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Abstract. Consider $(\mathcal{M},g)$ as a complete, simply connected Riemannian manifold. The aim of this paper is to provide various geometric estimates in different cases for the first eigenvalue of (p,q)-elliptic quasilinear system in both Dirichlet and Neumann conditions on Riemannian manifold. In some cases we add integral curvature condition and maybe we prove some theorems under other conditions.

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1 Introduction

Finding the bound of the eigenvalue for the Laplacian on a given manifold is a key aspects in Riemannian geometry. A major objective of this purpose is to study eigenvalue that appears as solutions of the Dirichlet or Neumann boundary value problems for curvature functions. In the recent years, because of the theory of self-adjoint operators, the spectral properties of linear Laplacian studied extensively. As an important example, mathematicians generally are interested in the spectrum of the Laplacian on compact manifolds with or without boundary or noncompact complete manifolds due to in these two cases the linear Laplacian can be uniquely extended to self-adjoint operators (see [1,2]).

Since the study of the properties of spectrum of Laplacian (specially in Dirichlet condition) in infinitely stretched regions has applications in elasticity, electromagnetism and

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quantum physics, it attracts attention of many mathematicians and physicists. Recently Mao has proved the existence of discrete spectrum of linear Laplacian on a class of 4-dimensional rotationally symmetric quantum layers, which are noncompact noncomplete manifolds in [3].

Consider M as a compact domain in a complete, simply connected Riemannian manifold $\mathcal{M}$. Let $u: M \longrightarrow \mathbb{R}$ be a smooth function on M or $u \in W^{1,p}(M)$ where $W^{1,p}(M)$ is the Sobolev space. The p-Laplacian of u for $1<p<\infty$ is defined as

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}(\mathrm{Hess}u)(\nabla u, \nabla u), \tag{1.1}$$

where

$$(\mathrm{Hess}u)(X,Y) = \nabla(\nabla u)(X,Y) = X(Yu) - (\nabla_X Y)u, \quad X,Y \in \chi(M).$$

Although the regularity theory of the p-Laplacian is very different from the usual Laplacian, many of the estimates for the first eigenvalue of the Laplacian (for example when $p=2$) can be generalized to general p. As an important example in [4], you can find remarkable results in a case of closed manifolds with bounded Ricci curvature by $(n-1)K$ where $K>0$. The special case $K=0$ and general case $K \in \mathbb{R}$ are studied in [5] and [6], respectively. Also you can find some similar topics in [7].

In a case of p-Laplacian also you can find various results in [8] in which Bakery-Emery curvature has a positive lower bound for weighted p-Laplacian and you can find useful results in [9] and [10] to general metric measure space.

Generally studying the eigenvalues of geometric operators are important tools for understanding the Riemannian manifolds. There are many mathematicians who work and give important results in a term of geometric quantities for the spectrum of Laplacian (you can see [11–13]).

1.1 The first eigenvalue of (p,q)-elliptic quasilinear system

In this paper we are going to obtain some geometric estimates for the first eigenvalue of (p,q)-elliptic quasilinear system which is defined as

$$\begin{cases} -\Delta_p u = \lambda u|u|^{p-2} + \lambda |u|^{\alpha}|v|^{\beta} v & \text{in } \mathcal{M}, \\ -\Delta_q v = \lambda v|v|^{q-2} + \lambda |u|^{\alpha}|v|^{\beta} u & \text{in } \mathcal{M}, \\ u = v = 0 \quad \text{(Dirichlet)} \quad \text{or} \quad \nabla_\delta u = \nabla_\delta v = 0 \quad \text{(Neumann)} & \text{on } \partial\mathcal{M}, \end{cases} \tag{1.2}$$

where δ is the outward normal vector on $\partial\mathcal{M}$, $p>1$, $q>1$ and α,β are real numbers such that

$$\alpha>0, \beta>0, \qquad \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$