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Dynamical Analysis on Traveling Wave of a Reaction-Advection-Diffusion Equation with Double Free Boundaries

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Abstract. This paper investigates a reaction-advection-diffusion equation with double free boundaries. The stationary solution of the system is studied by phase plane analysis. Then, the scale logarithm change sequence method is introduced to show the exact heteroclinic of the system with corresponding parameters. Moreover, a complete description of the types of traveling wave solutions is given with different advection term coefficients.

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Key Words: Phase plane analysis; stationary solution; heteroclinic orbit; traveling wave solution.

1 Introduction

Reaction-diffusion equation, a fundamental class of nonlinear partial differential equations, plays an important role in the fields of ecology, epidemiology, materials science, neural network etc. The classical reaction-diffusion equation

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t \geq 0, \tag{1.1}$$

was studied by scholars to describe the spreading of biological species or chemical substances [1–5]. Traveling wave solution, as a focal point in reaction-diffusion system, is

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regarded as the stationary solution of this system [6–9]. Wang and Xiong [10] proposed the explicit front wave solution of system

$$\begin{cases} u_t = du_{xx} + pu(1-u^\alpha)(u^\alpha + q), & d,p,\alpha \geq 0,\ q \in \mathbb{R}, \\ \lim\limits_{t\to-\infty} u(t,x) = 0, \quad \lim\limits_{t\to\infty} u(t,x) = 1. \end{cases} \tag{1.2}$$

In 2010, Du and Lin [11] estimated the solution u and the free boundary $h(t)$ corresponding to the equation (1.1) with Fisher-KPP type of nonlinearity and a free boundary. Furthermore, when $f(u)$ is a generalized Fisher-KPP type of nonlinearity

$$u_t = du_{xx} + ur(x-ct) - u^2, \qquad x \in \mathbb{R},\ t \geq 0, \tag{1.3}$$

Hu and Zou [12] achieved the traveling wave solutions and predicted the speed and manner of the extinction of a species in a shifting habitat representing a transition to a devastating environment.

In reality, due to the influence of various environmental factors, the migration and expansion of biological population show certain directionality, which can be described by the advection term in the reaction-diffusion equation. Therefore, many academics discussed the following equation

$$u_t = u_{xx} - \beta u_x + f(u), \qquad x \in \mathbb{R},\ t \geq 0. \tag{1.4}$$

In 2015, Gu et al. [13] researched the stationary solutions and traveling wave solutions of system (1.4) with a Fisher-KPP type of nonlinearity. Moreover, the traveling wave dynamical analysis on a specific case of equation (1.4)

$$u_t = u_{xx} - (a_0 + 2a_1 u)u_x + u(u^2-1)(u^2+\gamma), \qquad a_0, a_1 \in \mathbb{R},\ \gamma > 0, \tag{1.5}$$

has been studied in [14,15].

Inspired by the above works, we investigate the following free boundary problem

$$\begin{cases} u_t = u_{xx} - \beta u_x + u^m(1-u)(u-\alpha), & t>0,\ g(t)<x<h(t), \\ u(t,g(t)) = u(t,h(t)) = 0, & t>0, \\ g'(t) = -\mu u_x(t,g(t)), & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ h(0) = -g(0) = h_0, u(0,x) = u_0(x) & -h_0 \leq x \leq h_0, \end{cases} \tag{1.6}$$

where $\mu > 0$, $\beta \geq 0$, $0 < \alpha < 1$, $m \in \mathbb{N}^*$, $g(t)$, $h(t)$ are free boundaries to be determined. The initial function $u_0(x) \in \Phi(h_0)$ for $h_0 > 0$, and

$$\begin{aligned} \Phi(h_0) = \{\phi \in C^2([-h_0,h_0]) :\ & \phi(-h_0) = \phi(h_0) = 0,\ \phi'(-h_0) > 0,\ \phi'(h_0) < 0, \\ & \phi(x) > 0,\ x \in (-h_0,h_0)\}. \end{aligned}$$