JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **35**, No. 1, pp. 78-86

doi: 10.4208/jpde.v35.n1.5
March 2022

Blow-up of Classical Solutions to the Isentropic Compressible Barotropic Navier-Stokes-Langevin-Korteweg Equations

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Received 8 April 2021; Accepted 9 July 2021

Abstract. In this paper, we study the barotropic Navier-Stokes-Langevin-Korteweg system in $\mathbb{R}^3$. Assuming the derivatives of the square root of the density and the velocity field decay to zero at infinity, we can prove the classical solutions blow up in finite time when the initial energy has a certain upper bound. We obtain this blow up result by a contradiction argument based on the conservation of the total mass and the total quasi momentum.

AMS Subject Classifications: 35Q35, 35B44

Chinese Library Classifications: O175.29

Key Words: Navier-Stokes-Langevin-Korteweg system; classical solutions; blow up.

1 Introduction

We are concerned with the Cauchy problem for the isentropic compressible barotropic Navier-Stokes-Langevin-Korteweg system in $\mathbb{R}^3$:

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0, \\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla P+\mu\rho u=\dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)+\nu\operatorname{div}(\rho\mathbb{D}u), \\ \rho(0,x)=\rho_0(x), \quad u(0,x)=u_0(x), \end{cases} \tag{1.1}$$

where $\mathbb{D}u=(\nabla u+\nabla u^\top)/2$, the unknown functions $\rho:\mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}_+$ and $u:\mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}^3$ denote the density and the velocity field respectively, and $P:\mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}_+$ is the barotropic pressure of the form $P(\rho)=\rho^\gamma$ where $\gamma>1$ is the adiabatic constant. $\mu>0$,

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$\hbar > 0$ and $\nu > 0$ are the dissipation coefficient, the renormalized Plauck constant and the viscosity coefficient respectively.

The Navier-Stokes-Langevin-Korteweg equations include many classical equations, such as the compressible Navier-Stokes equations if $\mu = \hbar = 0$ which describe the law of mechanics of viscous fluids and are significant in fluid mechanics; the Navier-Stokes-Korteweg equations if $\mu = 0$ which are firstly considered by Van der Waal and Korteweg to model fluid capillarity effect and then developed by Dunn and Serrin [1] to reflect the variation of density at the interface of two phases; the Euler-Langevin-Korteweg equations if ν=0 which are firstly used as a stochastic interpretation of quantum mechanics [2], then applied to quantum semiconductor [3] and quantum trajectories of Bohmia mechanics [4], and recently have a renewed interest in statistical mechanics and cosmology [5].

There are many theoretical studies in blow up of smooth solutions of compressible Navier-Stokes equations. In 1998, Xin [6] showed a sufficient condition which leads to blow up of smooth solutions with initial density of compact support, and the key idea of proof is the total pressure decays faster in time in the presence of the vacuum. Then these results were improved by Xin and Wei [7] by showing the finite time blow up of smooth solutions with initial density containing vacuum and without the assumption of finite total energy. For the same questions, Lai [8] applied a contradiction argument to prove the classical solutions blow up in finite time, when the gradient of the velocity satisfies some decay constraint and the initial total momentum does not vanish. The blow up results of the full compressible case and the isentropic compressible case with constant and degenerate viscosities are obtained by Jiu [9], and a more precise blow up time can be computed out when the results are applied to Euler equations. Huang [10] presented a blow-up criterion for classical solutions in $\mathbb{R}^3$ in terms of the gradient of the velocity, which is the necessary condition of blow up. For the Navier-Stokes-Korteweg system, Tang [11] established a blow up result for the smooth solutions to the Cauchy problem of the symmetric barotropic case with initial density of compact support. The one-dimensional initial boundary value problem in a bounded domain was studied by Tang [12] and the blow up results of smooth solutions were obtained. Li [13] showed a Serrins type blow-up criterion for the density-dependent Navier-Stokes-Korteweg equations with vacuum in $\mathbb{R}^3$.

To our best knowledge, there isn't any blow up result for the Navier-Stokes-Langevin-Korteweg system yet and the dissipative term $\mu\rho u$ on the left side of $(1.1)_2$ brings some difficulties when we come to prove the total momentum is conserved. To overcome these difficulties, we define the total quasi momentum as

$$\int_{\mathbb{R}^3} e^{\mu t}\rho(t,x)u(t,x)\mathrm{d}x,$$

and we prove the classic solutions to the three-dimensional Navier-Stokes-Langevin-Korteweg system blow up in finite time by a contradiction argument based on the conservation of the total mass and the total quasi momentum in this paper.