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METRICALLY REGULAR MAPPING AND ITS UTILIZATION TO CONVERGENCE ANALYSIS OF A RESTRICTED INEXACT NEWTON-TYPE METHOD*

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Abstract

In the present paper, we study the restricted inexact Newton-type method for solving the generalized equation $0 \in f(x)+F(x)$, where X and Y are Banach spaces, $f : X \to Y$ is a Fréchet differentiable function and $F\colon X \rightrightarrows Y$ is a set-valued mapping with closed graph. We establish the convergence criteria of the restricted inexact Newton-type method, which guarantees the existence of any sequence generated by this method and show this generated sequence is convergent linearly and quadratically according to the particular assumptions on the Fréchet derivative of f. Indeed, we obtain semilocal and local convergence results of restricted inexact Newton-type method for solving the above generalized equation when the Fréchet derivative of f is continuous and Lipschitz continuous as well as $f + F$ is metrically regular. An application of this method to variational inequality is given. In addition, a numerical experiment is given which illustrates the theoretical result.

Mathematics subject classification: 47H04, 49J53, 65K10, 90C30.
Key words: Generalized equation, Restricted inexact Newton-type method, Metrically regular mapping, Partial Lipschitz-like mapping, Semilocal convergence.

1. Introduction

Let X and Y be Banach spaces, $f : X \to Y$ be a Fréchet differentiable function, and $F\colon X \rightrightarrows Y$ be a set-valued mapping with closed graph. In this paper, we are intended to find a point that satisfies the following generalized equation

$$0 \in f(x) + F(x). \tag{1.1}$$

The generalized equations of the type (1.1) were introduced by Robinson [48]. This type of generalized equation problem is an abstract model for a wide variety of variational problems including linear and nonlinear complementarity problems, systems of nonlinear equations, systems of inequalities and variational inequalities (see [48, 50] for more details). In particular, it may characterize optimality or equilibrium problems (see [24, 28] for more details).

The classical Newton-type method is one of the most important method for finding an approximate solution of (1.1), which was introduced by Dontchev [16] and defined as follows:

$$0 \in f(x_k) + \mathcal{D}f(x_k)(x_{k+1} - x_k) + F(x_{k+1}), \quad \text{for } k = 0, 1, \ldots, \tag{1.2}$$

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where x_0 is a given initial point and $\mathcal{D}f(x)$ is the derivative of f at x. When $F \equiv 0$, the above method reduces to the standard Newton method for solving the equation $f(x) = 0$ of the form:

$$f(x_k) + \mathcal{D}f(x_k)(x_{k+1} - x_k) = 0, \quad \text{for } k = 0, 1, \ldots.$$

There is a vast biographical research on inexact Newton-type methods for solving equation $f(x) = 0$ which employs different representation of inexactness; see for examples [7, 8, 54].

For solving generalized equation (1.1), Klatte and Kumer [32] generated a Newton sequence whose Newton steps are defined by approximations $f^{(k)}$ of f near the current iterate x_k and the solutions x_{k+1} of

$$0 \in f^{(k)}(x) + F(x),$$

and concentrated on local convergence analysis for Newton's method under certain type of approximations and different regularity conditions for $f + F$. Moreover, the authors [32, Proposition 8] have presented a Kantorovich-type statement, which is concerned on semilocal convergence, under pseudo-regularity of $f + F$ provided that all constructed Newton sequences are valid. Moreover, Aragon Artacho et al. [4] introduced Newton's iteration and presented its convergence analysis under metrically regular mapping when the single-valued part of the generalized equation (1.1) is an implicit function with a parameter. For solving generalized equation (1.1), a survey of local and semilocal convergence results for Newton's method can be found in [3, 5, 9, 16, 18, 19, 24, 37] and references therein.

In the case when the involved single-valued function f is not necessarily differentiable, we say that the generalized equation (1.1) is nonsmooth. The authors in [1] introduced a mapping $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ and applied a selection $\psi : X \to \mathcal{L}(X, Y)$ for $\mathcal{H}$ to the following method for solving nonsmooth generalized equation (1.1) and obtained a superlinear convergent result:

$$0 \in f(x_k) + \mathcal{H}(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \quad \text{for } k = 0, 1, 2, \ldots. \tag{1.3}$$

For solving nonsmooth generalized equation (1.1), Cibulka *et al.* [13] studied an inexact Newton method and obtained local convergence results for the method. Moreover, the semi-smooth Newton-type iterative procedure, for solving (1.1), was adopted by Cibulka *et al.* [14] (also see [24, Section 6F]). An extension of [33, Lemma 10.1] from equation to nonsmooth model (1.1) is given in [32, Theorem 4] via the concept of Newton maps [33]. Relevant results, for solving nonsmooth generalized equations (1.1), are given in [2, 6, 19, 27, 38, 39, 49, 52].

A large number of new developments on Newton methods with regularity properties of set-valued mappings for solving nonsmooth generalized equations have been studied in the last three decades and some of which have been accumulated in the monographs [17, 31, 33, 53].

Dembo et al. [15] introduced the following inexact Newton method for solving (1.1) with $F \equiv 0$, $X = Y = \mathbb{R}^n$ and f continuously differentiable with Jacobian ∇f in finite dimensional case:

$$(f(x_k) + \nabla f(x_k)(x_{k+1} - x_k)) \cap \mathbb{B}(0, \eta_k \|f(x_k)\|) \neq \emptyset, \tag{1.4}$$

where $\{\eta_k\} \subset (0, \infty)$ is a sequence of scalars and $\mathbb{B}(x, \alpha)$ denotes the closed ball centered at x with radius α.

Izmailov and Solodov [30] (see also in the monograph [31]) introduced the following inexact Newton method for solving the generalized equation (1.1) in the case of finite dimension:

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + e_k + F(x_{k+1}), \quad \text{where } e_k \in \mathbb{R}^n.$$