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A Stabilizer-Free Weak Galerkin Finite Element Method for the Stokes Equations

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Abstract. A stabilizer-free weak Galerkin finite element method is proposed for the Stokes equations in this paper. Here we omit the stabilizer term in the new method by increasing the degree of polynomial approximating spaces for the weak gradient operators. The new algorithm is simple in formulation and the computational complexity is also reduced. The corresponding approximating spaces consist of piecewise polynomials of degree $k\geq 1$ for the velocity and $k-1$ for the pressure, respectively. Optimal order error estimates have been derived for the velocity in both H^1 and L^2 norms and for the pressure in L^2 norm. Numerical examples are presented to illustrate the accuracy and convergency of the method.

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Key words: Stokes equations, weak Galerkin finite element method, stabilizer free, discrete weak differential operators.

1 Introduction

In this paper, we propose a stabilizer-free WG finite element method for the Stokes equations. For simplicity, we consider the Stokes equations with homogeneous Dirichlet boundary condition which seeks unknown vector-valued function $\mathbf{u}$ and scalar function p satisfying

$$-\Delta\mathbf{u}+\nabla p=\mathbf{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot\mathbf{u}=0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{u}=\mathbf{g} \qquad \text{on } \partial\Omega, \tag{1.1c}$$

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where Ω is a polygonal domain in $\mathbb{R}^d$ and $\mathbf{f}\in[L^2(\Omega)]^d$ is the unit external volumetric force acting on the fluid.

The variational formulation for the Stokes equations (1.1a)-(1.1c) is finding $\mathbf{u}\in[H^1(\Omega)]^d$ and $p\in L_0^2(\Omega)$ that satisfy $\mathbf{u}=\mathbf{g}$ on $\partial\Omega$ such that

$$(\nabla\mathbf{u},\nabla\mathbf{v})-(\nabla\cdot\mathbf{v},p)=(\mathbf{f},\mathbf{v}), \tag{1.2a}$$

$$(\nabla\cdot\mathbf{u},q)=0, \tag{1.2b}$$

for all $\mathbf{v}\in[H_0^1(\Omega)]^d$ and $q\in L_0^2(\Omega)$, where $H_0^1(\Omega)$ is defined in (2.1) and $L_0^2(\Omega)$ is defined as follows:

$$L_0^2(\Omega):=\Big\{q\in L^2(\Omega);\ \int_\Omega q dx=0\Big\}. \tag{1.3}$$

Various numerical methods have been developed for solving the Stokes equations, such as the finite element methods (FEMs) [3,7,8], the finite volume methods (FVMs) [5,27,28], and the finite difference methods [4,16,17]. Taking the classical conforming FEMs as an example, they are based on the variational form (1.2a)-(1.2b) and finite dimensional approximating subspaces of $[H_0^1(\Omega)]^d\times L_0^2(\Omega)$ consisting of piecewise polynomials. In those methods, the $\inf-\sup$ condition [1,2] has to be satisfied, which causes some limitations in constructing elements and generating meshes.

As a generalization of the classical conforming FEMs, the weak Galerkin (WG) finite element method has been gradually studied by researchers. This method was first introduced by Wang and Ye [12,18,19] for second order elliptic problems in 2013. Then it was extended to other partial differential equations (PDEs), such as Stokes problems [20,21], Brinkman problems [10,24,29], linear elasticity problems [22,23], biharmonic problems [11,13,14], and parabolic problems [6,31]. The WG method for the Stokes problems would adopt the following form: find $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}\in V_h$ and $p_h\in W_h$ satisfying $\mathbf{u}_h=Q_b\mathbf{g}$ on $\partial\Omega$ and

$$(\nabla_d\mathbf{u}_h,\nabla_d\mathbf{v}_h)-(\nabla_d\cdot\mathbf{v}_h,p_h)+s(\mathbf{u}_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_0), \tag{1.4a}$$

$$(\nabla_d\cdot\mathbf{u}_h,q_h)=0, \tag{1.4b}$$

for all $\mathbf{v}_h=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h$ and $q_h\in W_h$. Here V_h and W_h are the properly defined WG finite element spaces for the scalar variables and vector-valued variables, respectively. ∇_d is a weak gradient operator and $\nabla_d\cdot$ is a weak divergence operator to be detailed in Section 2. As a parameter free stabilizer, the bilinear form $s(\cdot,\cdot)$ enforces a certain weak continuity for the approximating solutions across element boundaries. In 2016, a new WG finite element method has been developed for solving the Stokes equations based on two gradient operators in [21]. This method employs two parameter independent stabilizers $s(\cdot,\cdot)$ and $c(\cdot,\cdot)$ for velocity functions and pressure functions, respectively. The corresponding numerical scheme is: find $\mathbf{u}_h\in\tilde{V}_h$ and $p_h\in\tilde{W}_h$ satisfying $\mathbf{u}_h=Q_b\mathbf{g}$ on $\partial\Omega$ such that

$$(\nabla_d\mathbf{u}_h,\nabla_d\mathbf{v})+(\mathbf{v}_0,\widetilde{\nabla}_d p_h)+s(\mathbf{u}_h,\mathbf{v})=(\mathbf{f},\mathbf{v}_0),$$

$$(\mathbf{u}_0,\widetilde{\nabla}_d q)-c(p_h,q)=0,$$