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Unconditional Optimal Error Estimates for the Transient Navier-Stokes Equations with Damping

Minghao Li[1], Zhenzhen Li[2] and Dongyang Shi[3,*]

[1] *College of Science, Henan University of Technology, Zhengzhou, Henan 450001, China*
[2] *College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, China*
[3] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China*

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Abstract. In this paper, the transient Navier-Stokes equations with damping are considered. Firstly, the semi-discrete scheme is discussed and optimal error estimates are derived. Secondly, a linearized backward Euler scheme is proposed. By the error split technique, the Stokes operator and the H^{-1}-norm estimate, unconditional optimal error estimates for the velocity in the norms $L^\infty(L^2)$ and $L^\infty(H^1)$, and the pressure in the norm $L^\infty(L^2)$ are deduced. Finally, two numerical examples are provided to confirm the theoretical analysis.

AMS subject classifications: 65N15, 65N30

Key words: Navier-Stokes equations with damping, linearized backward Euler scheme, error splitting technique, unconditional optimal error estimates.

1 Introduction

We consider the following transient Navier-Stokes equations with damping:

$$\begin{cases} \mathbf{u}_t-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\alpha|\mathbf{u}|^{r-2}\mathbf{u}+\nabla p=\mathbf{f} & \text{in } (0,T]\times\Omega, \\ \mathrm{div}\mathbf{u}=0 & \text{in } (0,T]\times\Omega, \\ \mathbf{u}=0 & \text{on } \partial\Omega, \\ \mathbf{u}(0,\cdot)=\mathbf{u}_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathcal{R}^2$ is a convex polygon domain with the boundary $\partial\Omega$, $\mathbf{u}=(u_1,u_2)$ and p are the fluid velocity and pressure, respectively, $\mathbf{f}$ is a given external force, ν is the viscosity coefficient, $2\le r<\infty$ and α are two damping parameters, respectively, and $|\cdot|$ is the

*Corresponding author.
Emails: lyminghao@126.com (M. Li), lizhenzhenpuyang@163.com (Z. Li), shi_dy@zzu.edu.cn (D. Shi)

Euclidean norm. The damping comes from the resistance to the motion of the flow. It describes various physical phenomena such as porous media flow, drag or friction effects, and some dissipative mechanisms [1,2].

The existence and uniqueness of global weak and strong solutions for the problem (1.1) were analyzed in [3,4]. At the same time, some studies have been devoted to the numerical analysis of the stationary incompressible Stokes or Navier-Stokes equations with damping. In [5], the MAC finite difference scheme was presented for the Stokes equations with damping on non-uniform grids. In [6], the conforming mixed finite element method (MFEM) was developed, and the existence and uniqueness of the weak solutions were proved. In [7], the superclose and superconvergence phenomenon of some stable MFEs were studied. In [8–10], the local projection stabilized MFEMs with the P_1-P_1 element pair were proposed for the Stokes or Navier-Stokes equations with damping. In [11, 12], the two-level and multi-level MFEMs were applied to the problem to save computation cost. In addition, the Navier-Stokes type variational inequality with nonlinear damping was also considered in [13]. However, there were few numerical methods reported for the transient Stokes or Navier-Stokes equations with damping.

On the other hand, in the finite element methods of the nonlinear problems, some time-step restrictions are usually required in the error estimates. In order to overcome this disadvantage, the error splitting technique was first presented in [14] for the nonlinear Joule heating equations and [15] for the incompressible miscible flow in porous media, respectively. Recently, this technique was applied to various nonlinear problems, such as the parabolic equation [16–18], the hyperbolic equation [19], the Schrödinger equation [20–24], the Landau-Lifshitz equation [25], the Ginzburg-Landau equations [26, 27], the Klein-Gordon-Schrödinger equations [28], the thermistor equations [29–31], the Navier-Stokes equations [32], the viscoelastic fluid flow equations [33], the MHD equations [34] and so on.

In this paper, we will research the transient Navier-Stokes equations with damping. We present the semi-discrete scheme for this problem, and derive optimal error estimates. Then we propose a linearized backward Euler scheme. Although unconditional optimal error estimates were obtained for the transient Navier-Stokes equations in [35, 36], the methods cannot be applied to the problem (1.1) for the nonlinear damping term may result in more complicated analysis, so we employ the error splitting technique in [32], which was used in the modified characteristics finite element methods of the Navier-Stokes equations. A time-discrete system is introduced, and the error is split into a temporal error and a spatial error. Then the temporal error and the regularity of the time-discrete system are presented. Subsequently, the space error and the boundedness of the velocity are derived. Finally, unconditional optimal error estimates are obtained. Especially, the analysis method of the pressure is different to that in [32]. The Stokes operator and the H^{-1}-norm estimate are employed, and consequently we obtained optimal error estimates for the pressure in the norm $L^\infty(L^\infty)$, while in [32], it is optimal in the norm $L^\infty(L^2)$.

Throughout this paper, we use the classical Sobolev spaces $W^{l,m}(\Omega)$, $L^l(\Omega)$, $H^m(\Omega)$