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Convergence Toward a Periodically Rotating One-Point Cluster in the Kinetic Thermodynamic Cucker-Smale Model

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Abstract. We study a relaxation dynamics of the kinetic thermodynamic Cucker-Smale (TCS) model under the effect of potential force field. For this, we present a sufficient framework on the emergence of a periodically rotating one-point cluster to the TCS model with a harmonic potential force. In the presence of external potential force, large-time behavior of the kinetic TCS model is completely different from that of the kinetic TCS model without an external force. For the relaxation toward the periodic motion in thermo-mechanical observables, we use dynamical systems theory such as the Lyapunov functional approach, method of characteristics and continuity argument. First, we derive the exponential decay estimate for the Lyapunov functionals of thermo-mechanical observables for the kinetic density f with a compact support. Second, we show that the supports of f in (x,v,θ) shrink to a periodically rotating one-point cluster exponentially fast. Our sufficient framework is formulated in terms of initial configuration, coupling strengths and communication weight functions.

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Key words: One-point cluster, harmonic potential field, thermodynamic Cucker-Smale ensemble, kinetic model.

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1 Introduction

The purpose of this paper is to continue previous studies in [7] on the formation of one-point cluster in the thermodynamic Cucker-Smale ensemble under the effect of a harmonic potential field. Collective behaviors of many-body systems are often observed in nature, to name a few, aggregation of bacteria [25], flashing of fireflies [3, 27], flocking of birds [26], schooling of fish [11, 22, 24] and synchronous firing of pacemaker cells [21], etc. For the brief introduction of the subject, we refer to survey articles [1, 2, 20]. To provide a realistic model for the collective behaviors of active particles, we need to introduce a dynamical system taking into account not only interactions between mechanical observables (position and velocity) but also extra interactions due to the internal observables such as temperature and spins, etc. As a first step toward the realistic flocking modeling of active particles, Ha and Ruggeri introduced a generalized Cucker-Smale model [18] for thermo-mechanical ensemble, so called the thermodynamic Cucker-Smale model. In this work, we further investigate the effect of an external potential force field on the emergent dynamics of the thermodynamic Cucker-Smale ensemble.

Let x_α, v_α and θ_α be the position, velocity and temperature of the α-th TCS particle, respectively. Then, their temporal dynamics is governed by the Cauchy problem to the following (particle) TCS model:

$$\begin{cases} \dfrac{dx_\alpha}{dt} = v_\alpha, \quad t>0, \quad \alpha \in [N] := \{1,\dots,N\}, \\ \dfrac{dv_\alpha}{dt} = \dfrac{\kappa_1}{N}\sum\limits_{\beta=1}^{N} \psi(|x_\alpha - x_\beta|)\left(\dfrac{v_\beta - \bar{v}}{\theta_\beta} - \dfrac{v_\alpha - \bar{v}}{\theta_\alpha}\right) - \nabla_{x_\alpha} U(x_\alpha), \\ \dfrac{d}{dt}\left(\theta_\alpha + \dfrac{1}{2}|v_\alpha|^2\right) = \dfrac{\kappa_2}{N}\sum\limits_{\beta=1}^{N} \zeta(|x_\alpha - x_\beta|)\left(\dfrac{1}{\theta_\alpha} - \dfrac{1}{\theta_\beta}\right) \\ \qquad\qquad\qquad\qquad + \dfrac{\kappa_1}{N}\sum\limits_{\beta=1}^{N} \psi(|x_\alpha - x_\beta|)\left(\dfrac{v_\beta - \bar{v}}{\theta_\beta} - \dfrac{v_\alpha - \bar{v}}{\theta_\alpha}\right)\cdot \bar{v}, \end{cases} \tag{1.1}$$

subject to initial data

$$(x_\alpha, v_\alpha, \theta_\alpha)(0) = \left(x_\alpha^0, v_\alpha^0, \theta_\alpha^0\right),$$

where κ_1 and κ_2 denote nonnegative coupling strengths, and the ensemble averages $(\bar{x}, \bar{v})$ are given as follows:

$$\bar{x} := \frac{1}{N}\sum_{\alpha=1}^{N} x_\alpha, \quad \bar{v} := \frac{1}{N}\sum_{\alpha=1}^{N} v_\alpha. \tag{1.2}$$