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# Surface Embedding of Non-Bipartite $k$-Extendable Graphs

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**Abstract.** For every surface, we find the minimum number $k$ such that every non-bipartite graph that is embeddable in that surface is not $k$-extendable. In particular, we construct a family of 3-extendable graphs which we call bow-tie graphs. This confirms the existence of an infinite number of 3-extendable non-bipartite graphs that are embeddable in the Klein bottle.

**AMS subject classifications**: 05C10, 05C70

**Key words**: Non-bipartite graph, matching extension, surface embedding.

## 1 Introduction

A matching $M$ of a graph $G$ is said to be extendable if $G$ has a perfect matching containing $M$. A graph is $k$-extendable if it has a matching consisting of $k$ edges and every matching consisting of $k$ edges is extendable, where

$$1 \le k \le (|V(G)|-2)/2.$$

Much attention to the theory of matching extension has been paid since it was introduced by Plummer [17] in 1980. We recommend Lovász and Plummer's book [11] for

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an excellent survey of the matching theory, and [21,27] for recent progress. Interests in the matching extensions of graphs embedded on surfaces began with the charming result [19] that no planar graph is 3-extendable. We refer the reader to Gross and Tucker's book [5] for basic notions on topological graph theory; see also [2].

Plummer [18] considered the problem of determining the minimum integer $k$ such that every $\Sigma$-embeddable graph is not $k$-extendable. Based on some partial results of Plummer, Dean [3] found the complete answer to this problem.

**Theorem 1.1** (Dean, Plummer)**.** *Let $\Sigma$ be a surface of characteristic $\chi$. Let $\mu(\Sigma)$ be the minimum integer $k$ such that every $\Sigma$-embeddable graph is not $k$-extendable. Then we have*

$$\mu(\Sigma)=\begin{cases}3, & \text{if } \chi=2,\\ 2+\lfloor\sqrt{4-2\chi}\rfloor, & \text{otherwise.}\end{cases} \tag{1.1}$$

Its proof made a heavy use of the Euler contribution technique, which dates back to Lebesgue [8], developed by Ore [14], and flourished by Ore and Plummer [15].

In a previous paper [12], we extended Theorem 1.1 by finding the minimum integer $k$ such that there is no $\Sigma$-embeddable $(n,k)$-graphs, where an $(n,k)$-graph is a graph whose subgraph obtained by removing any $n$ vertices is $k$-extendable. This paper continues the study of this embeddable-extendable type of problems. We dig a little deeper by concentrating on non-bipartite graphs. Here is our main result.

**Theorem 1.2.** *Let $\Sigma$ be a surface of characteristic $\chi$. Let $\mu'(\Sigma)$ to be the minimum integer $k$ such that every $\Sigma$-embeddable non-bipartite graph is not $k$-extendable. Then we have*

$$\mu'(\Sigma)=\begin{cases}4, & \text{if } \chi\in\{-1,0\},\\ 3, & \text{if } \chi=2,\\ \lfloor(7+\sqrt{49-24\chi})/4\rfloor, & \text{otherwise.}\end{cases} \tag{1.2}$$

Non-bipartite graphs differ from bipartite graphs in many aspects, even if we are concerned with only matching problems. For instance, König theorem states that the maximum size of a matching in a bipartite graph equals the minimum size of a vertex cover; see Rizzi [24] for a short proof. Taking a triangle as the graph under consideration, one may see immediately that non-bipartite graphs do not admit this beautiful property in general.

Another example is on the algorithmic complexity. Lakhal and Litzler [7] discovered a polynomial-time algorithm for the problem of finding the extendability of a bipartite graph. It is still unknown that whether the same extendability problem for non-bipartite graphs can be solved in polynomial time or not; see Plummer [20].