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Discrete Duality Finite Volume Discretization of the Thermal-P_N Radiative Transfer Equations on General Meshes

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Abstract. The *discrete duality finite volume* method has proven to be a practical tool for discretizing partial differential equations coming from a wide variety of areas of physics on nearly arbitrary meshes. The main ingredients of the method are: (1) use of three meshes, (2) use of the Gauss-Green theorem for the approximation of derivatives, (3) discrete integration by parts. In this article we propose to extend this method to the coupled grey thermal-P_N radiative transfer equations in Cartesian *and* cylindrical coordinates in order to be able to deal with two-dimensional Lagrangian approximations of the interaction of matter with radiation. The stability under a Courant-Friedrichs-Lewy condition and the preservation of the diffusion asymptotic limit are proved while the experimental second-order accuracy is observed with manufactured solutions. Several numerical experiments are reported which show the good behavior of the method.

AMS subject classifications: 65N06, 65N08, 65N12

Key words: Hybrid multiscale models, radiative transfer equation, grey P_N approximation, discrete duality finite volume method.

1 Introduction

For discretizing the Boltzmann equation under its multigroup P_N approximate form we proposed in [1] a DDFV (Discrete Duality Finite Volume) type method which benefits from several attractive features, namely: use of general meshes, second-order experimental accuracy on a broad variety of meshes, stability under a Courant-Friedrichs-Lewy (CFL) condition, preservation of the diffusion asymptotic limit and generalization from

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*Corresponding author. *Email addresses:* `francois.hermeline@cea.fr` (F. Hermeline) rectangle meshes to nearly arbitrary meshes of either legacy methods as the MAC [2] and Yee's schemes [3] or the more recent STARMAP method [4].

This was considered as a first step before tackling the discretization of the coupling of the hydrodynamics and Boltzmann equations, thus allowing the Lagrangian type methods to be used for the hydrodynamics without the possible mesh distortions due to the matter motion being an inconvenient for the approximation of the Boltzmann equation. Here we propose to address a second step namely the discretization of the coupled thermal-Boltzmann equations on general meshes (without heat conduction to simplify) while the background material is assumed to be static. Furthermore we deal with both Cartesian and cylindrical coordinates. Of course the third (and last) step will be the discretization of the whole coupled hydrodynamics-Boltzmann equations that is to say the coupling not only of the thermal equation but also the mass and momentum conservation equations with the Boltzmann equation. Such a step will be the subject of ongoing works.

To our knowledge designing numerical methods combining all the features cited above is a rather challenging task: see [5, 6] where a Godunov-type method is proposed for solving the hyperbolic heat equation and, more generally, the abstract *Friedrichs* systems of equations on unstructured meshes. See also [7] where a discontinuous Galerkin finite element method is proposed for the coupled thermal-Boltzmann equations in Cartesian coordinates.

We will focus here on the Boltzmann equation for *photons*, namely the *radiative transfer* equation, but of course other type of particles (as neutrons for example) could be dealt with as well.

The organization of the paper is as follows. In Section 2 we recall what are the thermal-radiative transfer equations and their grey approximate form and what are the grey coupled thermal-P_N radiative transfer equations in both Cartesian and cylindrical coordinates. The finite volume method we propose is described in Section 3 in the two-dimensional cylindrical coordinates framework (we restrict ourselves to the 2D case for simplicity but the ideas behind the discretization can be generalized to the 3D case although the notation is inherently more intricate). Section 3.1 is devoted to the DDFV discretization of partial derivatives on general meshes and to some of its mimetic properties. Sections 3.2 and 3.3 detail respectively the space and time discretizations of the P_N radiative transfer equations. In Section 3.4 the L^2 stability of the scheme is studied (Theorems 3.1 to 3.3) while we prove that it is asymptotic preserving (Theorem 3.4). Section 4 details the space and time discretizations of the grey thermal-P_N radiative transfer equations. Finally three numerical experiments are presented in Section 5, the first one allowing us to assess the experimental order of convergence with a manufactured solution and the others being reference benchmarks inspired from the literature [8, 11, 12]. Concluding remarks are proposed in Section 6.

In all what follows the vectors and the matrices will be noted with bold letters while $\boldsymbol{x}=(x,y,z)$ or $\boldsymbol{x}=(r,\phi,z)$, R, S^2, t, δ_m^n, $\boldsymbol{I}_n$ and i will denote respectively the position vector in Cartesian or cylindrical coordinates, the set of real numbers, the unit sphere in R^3, the