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An Error Estimate of a Modified Method of Characteristics Modeling Advective-Diffusive Transport in Randomly Heterogeneous Porous Media

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Abstract. We analyze a stochastic modified method of characteristics (MMOC) modeling advective-diffusive transport in randomly heterogeneous porous media. Under the log-normal assumption of the porous media and the finite-dimensional noise assumption that leads to unbounded diffusivity, we prove an optimal-order error estimate for the stochastic MMOC scheme. Numerical experiments are presented to substantiate the numerical analysis.

AMS subject classifications: 65M25, 65M60, 65Z05, 76M10

Key words: Uncertainty quantification, MMOC, advective-diffusive transport.

1 Introduction

The objective in many applications such as remediation of contaminated aquifers, miscible displacement in enhanced oil recovery, and CO_2 sequestration is to accurately predict the moving steep fronts of the concentration of the solute or injected solvent to optimize the remediation or recovery process [4, 11, 20, 26, 35]. Ideally, with the given information on the media (e.g., permeability and porosity) and fluid (e.g., the pressure and the concentration of the solute or solvent at the injection wells or sources of the contaminations and at the production wells or monitoring wells), one should be able to determine the movement of the solute or solvent. However, many very difficult mathematical and numerical obstacles occur. The mapping from the given data to the concentration of the solute/solvent is a strongly coupled, nonlinear and dynamic process, in which the

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transport of the solute/solvent depends heavily on the Darcy velocity of the fluid while the viscosity of the fluid may in turn depend on the concentration of solvent. Furthermore, information on subsurface porous media is very limited and available only near the injection and production (or monitoring) wells. Due to the high spatial variability of geological properties of porous media and the scarcity of available data, the hydrological parameters describing macroscopic properties of porous media, such as the intrinsic permeability and porosity, cannot be accurately characterized in detail and are often modeled as spatially correlated random fields [7,16,45]. These parameters strongly affect the transport processes, thus rendering the transport processes uncertain and making the flow and transport equations stochastic. In summary, the mathematical models lead to strongly coupled nonlinear systems of time-dependent advection-diffusion equations that present moving steep fronts, where complex physical and chemical phenomena take place that need to be resolved accurately in applications [4,11,20,27].

Upwind methods are widely used in industrial applications to stabilize the numerical approximations to these systems in large-scale simulators, but they tend to produce numerical solutions with excessive numerical diffusion and spurious grid orientation effects [11,33]. On the other hand, Eulerian-Lagrangian methods [2,8,9,12,14,23,31–33,37,39] combine the advection term with the capacity term in the transport equation and carry out the temporal discretization through a characteristic tracking. They symmetrize the transport equation and yield a symmetric and positive definite linear algebraic system. Moreover, they naturally cancel out the majority of the temporal error in the transport equation, which most Eulerian methods attempt to reduce via different techniques, by the spatial error from the advection term. Therefore, Eulerian-Lagrangian methods generate accurate numerical solutions even if large time steps and coarse spatial grids are used, and are very competitive with many numerical methods [33, 34]. Furthermore, they eliminate the excessive numerical diffusion and grid orientation effect present in upstream-weighted, large-scale numerical simulators in industrial production [11,36].

In this paper we develop a stochastic MMOC method, an MMOC-based stochastic Galerkin method [13,44,46], for a time-dependent advection-diffusion equation modeling solute transport in randomly heterogeneous porous media. We follow the treatment in [3, 18, 28, 30, 42, 43, 45] to make a physically relevant assumption that the diffusivity coefficient is log-normal, which is unbounded and violates the conventional assumption that the diffusivity has uniform lower and upper bounds, cf. e.g., [3, Equation 5.2] and [5, Equation 1.1], that is crucial in the corresponding analysis. Furthermore, the spatial eigenfunctions and the random variables are coupled in the Karhunen-Loéve expansion in the random diffusivity coefficient. Thus the developed numerical analysis frameworks for the MMOC to deterministic problems (cf. e.g., [8,9]), which are based on the boundedness of the diffusivity coefficients, do not directly apply. Besides, due to the dependence of the diffusivity coefficient on stochastic variables, which will change their original joint density function, some estimates of the projection in parametric space are affected and require unconventional treatments (cf. Lemma 5.2).

The rest of the paper is organized as follows. In Section 2 we formulate the mathe-