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ADAPTIVE MULTIGRID METHOD FOR EIGENVALUE PROBLEM

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Abstract. In this paper, we propose a type of adaptive multigrid method for eigenvalue problem based on the multilevel correction method and adaptive multigrid method. Different from the standard adaptive finite element method applied to eigenvalue problem, with our method we only need to solve a linear boundary value problem on each adaptive space and then correct the approximate solution by solving a low dimensional eigenvalue problem. Further, the involved boundary value problems are solved by some adaptive multigrid iteration steps. The proposed adaptive algorithm can reach the same accuracy as the standard adaptive finite element method for eigenvalue problem but evidently reduces the computational work. In addition, the corresponding convergence and optimal complexity analysis are derived theoretically and numerically, respectively.

Key words. Eigenvalue problem, adaptive multigrid method, multilevel correction, convergence, optimal complexity.

1. Introduction

How to solve large-scale eigenvalue problems is a very significant problem in modern scientific and engineering calculations. Many physical models and engineering models ultimately boil down to eigenvalue problems, such as the structural vibration analysis in buildings design, stability analysis in control systems, inherent frequency analysis of aircraft, etc. In recent years, the first-principles electronic structure calculations have pushed into the spotlight, and its key point is right to solve a class of nonlinear eigenvalue models. Therefore, it is necessary to make an indepth study of eigenvalue problem for its important theoretical significance and wide application value.

Among different numerical methods for eigenvalue problems, the adaptive finite element method (AFEM) is an efficient approach in generating optimal triangulation. AFEM was proposed by Babuška and his cooperators in [4, 5]. Up to now, the corresponding theoretical analysis of AFEM is well-developed. The convergence and optimal complexity analysis for boundary value problem can be found in [10, 16, 22, 23, 34, 33, 35, 37]. For eigenvalue problems, we can also find some similar results in [15, 17, 18, 19, 27, 30]. To further improve the efficiency of adaptive finite element method, the multilevel technique was absorbed to generate the adaptive multigrid method. Actually, it is worthing noting that adaptive mesh refinement technique was confirmed fully compatible with the multilevel mesh structure. Based on this idea, Brandt [6, 8] introduced the multilevel adaptive technique (MLAT), and McCormick [31] developed the fast adaptive composite grid method (FAC). For more results about the adaptive multigrid method, please refer to [13, 21, 32, 38, 39] and the references cited therein.

Though the optimal triangulations can be derived by standard AFEM, we have to solve an eigenvalue problem on each adaptive space, which is time-consuming

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and very tedious with the growth of degree of freedoms. The purpose of this paper is to propose a new type of adaptive multigrid method for solving eigenvalue problem based on adaptive finite element method, adaptive multigrid method and the recent work on the multilevel correction method [12, 24, 26, 28, 29, 40, 41]. In addition, we also analyze the corresponding convergence and optimal complexity property. In our presented adaptive multigrid method, the eigenvalue problem can be transformed into a series of linear boundary value problems on the fine grids and some eigenvalue problems on the coarsest grid. The dimension of the small-scale eigenvalue problem will be fixed during the adaptive refinement, thus the solving time can be ignored if the size of mesh becomes increasingly smaller after some refinement steps. Further, for the involved linear boundary value problems, we only need to proceed some multigrid iteration steps on the newly refined elements and their neighbors. For more details, please refer to [6, 21, 38, 39] and references cited therein. In this paper, we will adopt the techniques in [10, 15, 22] to prove the convergence and optimal complexity of the proposed adaptive multigrid method.

The rest of the paper is arranged as follows. Section 2 describes some basic notations and the standard AFEM for the second order elliptic boundary value problem. In Section 3, we introduce the adaptive multigrid method for eigenvalue problems. The corresponding convergence and complexity analysis will be given in Section 4. In Section 5, some numerical experiments are presented to validate the efficiency and the theoretical analysis. Section 6 concludes.

2. Preliminaries of standard adaptive finite element method for boundary value problem

This section is devoted to introducing some basic notation and some useful results of AFEM for second order linear elliptic boundary value problems. We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with associated norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ (see, e.g., [1]). For $p=2$, we denote $H^s(\Omega)=W^{s,2}(\Omega)$, $H_0^1(\Omega)=\{v\in H^1(\Omega):\ v|_{\partial\Omega}=0\}$, where $v|_{\partial\Omega}=0$ is in the sense of trace and $\|\cdot\|_{s,\Omega}=\|\cdot\|_{s,2,\Omega}$. For simplicity, we set $V=H_0^1(\Omega)$ in the rest of this paper.

Here, we consider the following homogeneous boundary value problem:

$$\left\{\begin{array}{rcll} Lu := -\nabla\cdot(\mathcal{A}\nabla u)+\phi u &=& f & \text{in } \Omega,\\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

where $\mathcal{A}=(a_{i,j})_{d\times d}$ is a symmetric positive definite matrix with suitable regularity and ϕ is a nonnegative function.

In order to use the finite element method, we first introduce the weak form for (1) as follows: Find $u\in V$ such that

$$a(u,v)=(f,v),\quad \forall v\in V, \tag{2}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v)=\int_\Omega(\mathcal{A}\nabla u\cdot\nabla v+\phi uv)d\Omega. \tag{3}$$

Obviously, the bilinear form $a(\cdot,\cdot)$ is bounded and coercive over V. Thus, we can define the energy norm $\|\cdot\|_{a,\Omega}$ by $\|w\|_{a,\Omega}=\sqrt{a(w,w)}$.

Now, we introduce the standard finite element method for linear boundary value problem (2). Firstly, we generate a shape regular decomposition of the computing domain $\Omega\subset\mathcal{R}^d$ $(d=2,3)$ into triangles or rectangles for $d=2$, tetrahedrons or