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Application of the Alternating Direction Method of Multipliers to Control Constrained Parabolic Optimal Control Problems and Beyond

Roland Glowinski[1,2], Yongcun Song[3], Xiaoming Yuan[4,*] and Hangrui Yue[5]

[1] *Department of Mathematics, University of Houston, 44800 Calhoun Road, Houston, TX 77204, USA*
[2] *Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong, China*
[3] *Chair for Dynamics, Control and Numerics, Alexander von Humboldt-Professorship, Department of Data Science, Friedrich-Alexander-Universität Erlangen-Nürnberg, 91058 Erlangen, Germany*
[4] *Department of Mathematics, The University of Hong Kong, Hong Kong, China*
[5] *School of Mathematical Sciences, Nankai University, Tianjin 300071, China*

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Abstract. Control constrained parabolic optimal control problems are generally challenging, from either theoretical analysis or algorithmic design perspectives. Conceptually, the well-known alternating direction method of multipliers (ADMM) can be directly applied to such problems. An attractive advantage of this direct ADMM application is that the control constraints can be untied from the parabolic optimal control problem and thus can be treated individually in the iterations. At each iteration of the ADMM, the main computation is for solving an unconstrained parabolic optimal control subproblem. Because of its inevitably high dimensionality after space-time discretization, the parabolic

[*]Corresponding author.
Emails: `roland@math.uh.edu` (R. Glowinski), `ysong307@gmail.com` (Y. Song), `xmyuan@hku.hk` (X. Yuan), `yuehangrui@gmail.com` (H. Yue)

optimal control subproblem at each iteration can be solved only inexactly by implementing certain numerical scheme internally and thus a two-layer nested iterative algorithm is required. It then becomes important to find an easily implementable and efficient inexactness criterion to perform the internal iterations, and to prove the overall convergence rigorously for the resulting two-layer nested iterative algorithm. To implement the ADMM efficiently, we propose an inexactness criterion that is independent of the mesh size of the involved discretization, and that can be performed automatically with no need to set empirically perceived constant accuracy a priori. The inexactness criterion turns out to allow us to solve the resulting parabolic optimal control subproblems to medium or even low accuracy and thus save computation significantly, yet convergence of the overall two-layer nested iterative algorithm can be still guaranteed rigorously. Efficiency of this ADMM implementation is promisingly validated by some numerical results. Our methodology can also be extended to a range of optimal control problems modeled by other linear PDEs such as elliptic equations, hyperbolic equations, convection-diffusion equations, and fractional parabolic equations.

AMS subject classifications: 49M41, 35Q90, 35Q93, 65K05, 65K10, 90C25

Key words: Parabolic optimal control problem, control constraint, alternating direction method of multipliers, inexactness criterion, nested iteration, convergence analysis.

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1 Introduction

Optimal control problems modeled by partial differential equations (PDEs) with additional constraints on the control and/or state variables capture important applications in various areas, such as physics, chemistry, engineering, medicine, and financial engineering. We refer to, e.g., [21–23,33,37,55], for a few references. These problems are generally difficult from either theoretical analysis or algorithmic design perspectives; one reason being that the optimal control problems and other constraints on the control and/or state variables are coupled. The high dimensionality of the resulting algebraic systems after discretization further explains the lack of a rich set of efficient numerical methods in the literature, especially for some optimal control problems modeled by time-dependent PDEs. To tackle such a problem numerically, a general principle is that the structures and properties of the model should be sophisticatedly considered in algorithmic design, rather than applying some existing algorithms generically. One particular desire is to untie the optimal control problem and other constraints on the control and/or state variables, and treat them individually in the iterations.