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Gorenstein Hereditary Rings

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Abstract. In this paper, we introduce the Gorenstein hereditary and Gorenstein semi-hereditary modules and study the properties of the radical of Gorenstein projective modules over left artin rings. As an application, we characterize when Gorenstein projective modules are projective and give some characterizations of the Gorenstein hereditary rings by radical and Gorenstein projective modules over the endomorphism algebra of a module, respectively. Meanwhile we study the projective complexes over Gorenstein hereditary rings. In the last part of the paper, we study the heredity, Gorenstein heredity and quasi-heredity of the Morita context ring $\Lambda_{(0,0)}$ which is a left artin algebra.

AMS subject classifications: 16E60, 16N20, 16S50, 16D10

Key words: Gorenstein hereditary ring, Gorenstein projective module, radical, projective complex, Morita ring.

1 Introduction

Let R be a ring with an identity element and all modules are unitary. Recall from [7] that an R-module M is called Gorenstein projective if there exists an exact sequence

$$\cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots$$

of projective R-modules with $M \cong \mathrm{Im} d^{-1}$ such that $\mathrm{Hom}_R(-,Q)$ leaves the sequence exact whenever Q is a projective R-module. An R-module M is called Gorentein flat [8] if there exists an exact sequence

$$\cdots \longrightarrow F^{-1} \xrightarrow{f^{-1}} F^0 \xrightarrow{f^0} F^1 \xrightarrow{f^1} \cdots$$

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of flat R-modules with $M\cong \mathrm{Im} f^{-1}$ such that $E\otimes_R -$leaves the sequence exact whenever E is an injective R-module. The Gorenstein projective and flat dimensions of an R-module M are defined in terms of Gorenstein projective and flat resolutions, and denoted by $\mathrm{Gpd}_R(M)$ and $\mathrm{Gfd}_R(M)$, respectively, see [15]. As in the literature [3], the Gorenstein global dimension of R is

$$\mathrm{Ggl.dim}(R)=\sup\{\mathrm{Gpd_R(M)}|M \text{ is an R-module}\}.$$

In [4], the authors studied particular cases of Gorenstein projective and flat modules, which they called strongly Gorenstein projective and flat modules, respectively. Recall that an R-module M is said to be strongly Gorenstein projective if there exists an exact sequence

$$\cdots \longrightarrow P \xrightarrow{d} P \xrightarrow{d} P \xrightarrow{d} \cdots$$

of projective R-modules with $M\cong \mathrm{Im} d$ such that $\mathrm{Hom}_R(-,Q)$ leaves the sequence exact whenever Q is a projective R-module. Every projective module is strongly Gorenstein projective, every strongly Gorenstein projective module is Gorenstein projective. An R-module M is said to be strongly Gorenstein flat if there exists an exact sequence

$$\cdots \longrightarrow F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

of flat R-modules with $M\cong \mathrm{Im} f$ such that $E\otimes_R -$leaves the sequence exact whenever E is an injective R-module. Every flat module is strongly Gorenstein flat, every strongly Gorenstein flat module is Gorenstein flat.

In [20], Shrikhande investigated the hereditary and semi-hereditary modules. Recall that an R-module is called hereditary if every R-submodule of it is projective, and an R-module M is called semi-hereditary if every finitely generated submodule of M is projective. A ring is called hereditary in case each ideal is projective. In this paper, we introduce and study Gorenstein hereditary and Gorenstein semi-hereditary modules.

The class of Morita rings plays an important role in ring theory and covers many special rings, and has been extensively studied (see, e.g., [11, 13, 16, 23]). Formal triangular matrix rings are some of the obvious examples of Morita contexts. Quasi-hereditary algebras were introduced by Scott [21] to study highest weight categories in the representation theory of semisimple complex Lie algebras and algebraic groups. Cline, Parshall and Scott proved many important results in [5] and [18]. Zhu in [29] proved the quasi-heredity of the triangular matrix algebras of quasi-hereditary algebras A and B by a bimodule ${}_AM_B$ under a suitable condition on the bimodule M and described the good module category over this quasi-hereditary triangular matrix algebra and the characteristic module of it. In this paper, we study the quasi-heredity of the Morita algebra $\Lambda_{(0,0)}$ with zero bimodule homomorphisms.

This paper is organized as follows. In Section 2, we introduce and study Gorenstein hereditary and Gorenstein semi-hereditary modules. In Section 3, we study the properties of the radical of Gorenstein projective modules over left artin rings. As an application, we characterize when Gorenstein projective modules are projective and give some