Ann. of Appl. Math.
33:3(2017), 266-288

A CANONICAL CONSTRUCTION OF H^m-NONCONFORMING TRIANGULAR FINITE ELEMENTS[*†]

Jun Hu[‡]
(*LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, PR China*)
Shangyou Zhang
(*Dept. of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA*)

Abstract

We design a family of 2D H^m-nonconforming finite elements using the full P_{2m-3} degree polynomial space, for solving $2m$th elliptic partial differential equations. The consistent error is estimated and the optimal order of convergence is proved. Numerical tests on the new elements for solving tri-harmonic, 4-harmonic, 5-harmonic and 6-harmonic equations are presented, to verify the theory.

Keywords nonconforming finite element; minimum element; high order partial differential equation

2000 Mathematics Subject Classification 65N30; 73C02

1 Introduction

For solving $2m$th order elliptic partial differential equations, the finite element spaces are designed as either a subspace of H^m Sobolev space, or not a subspace. In the first case, the finite element is called a conforming element. In the latter case, the finite element is called a non-conforming element. But some continuity is still required for non-conforming finite elements. The Courant triangle, the space of continuous piecewise linear functions, is an H^1 conforming finite element, solving second order elliptic equations. The Crouzeix-Raviart triangle, the space of piecewise linear functions continuous at mid-edge points of each triangle, is a P_1 H^1-nonconforming finite element. The possible minimum polynomial degree is m for an H^m conforming

[*]The research of the first author was supported by NSFC projection 11625101, 91430213 and 11421101.

[†]Manuscript received October 14, 2016

[‡]Corresponding author. E-mail: hujun@math.pku.edu.cn

and non-conforming finite element. This is because an mth order derivative of polynomial degree $m-1$ or less would be zero. Wang and Xu constructed a family of P_m nonconforming finite elements for $2m$th-order elliptic partial differential equations in R^n for any $n \geq m$, on simplicial grids [18]. Such minimum finite elements are very simple compared with the standard conforming elements. For example, in 3D, for $m = 2, 3, 4$ the polynomial degrees of the H^2, H^3 and H^4 elements are 9, 17 and 25, respectively, cf. [1, 2, 20], while those of Wang-Xu's elements are 2, 3 and 4 only, respectively. However, there is a limit that the space dimension n must be no less than the Sobolev space index m. For example, Wang and Xu constructed a P_3 H^3-nonconforming element in 3D [18], but not in 2D.

On rectangular grids, the problem of constructing H^m conforming elements is relatively simple. Hu, Huang and Zhang constructed an n-D C^1-Q_2 element on rectangular grids [10]. Here Q_k means the space of polynomials of separated degree k or less. Then, the element is extended to a whole family of C^{k-1}-Q_k elements, i.e., H^k-conforming Q_k elements for any space dimension n, in [11]. That is, the minimum polynomial degree k $(= m)$ is achieved in constructing H^m-conforming finite elements, on rectangular grids for any space dimension n. There is no limit of Wang-Xu [18] that $n \geq m$.

It is a challenge to remove the limit $n \geq m$ in the Wang-Xu's work [18], by constructing the minimum degree non-conforming H^m finite elements for the space dimension $n < m$. First, in 2D, we need to construct H^m non-conforming finite elements of polynomial degree m on triangular grids, $m > 2$. This is not possible on general grids. In [12] Hu-Zhang constructed an H^3 non-conforming finite element of cubic polynomials, but on the Hsieh-Clough-Tocher macro-triangle grids, following the idea in the construction of H^m conforming elements on macro rectangular grids in [10,11]. In [19], Wu-Xu enriched the P_3 polynomial space by 3 P_4 bubble functions to obtain a working H^3 non-conforming element in 2D. In fact, they extended this technique to n space dimension [19] so that H^{n+1} non-conforming elements in n space dimension is constructed by P_{n+1} polynomials enriched by n P_{n+2} face-bubble functions. In this work, we use the full P_{2m-3} polynomial space for $m \geq 4$ to construct 2D H^m non-conforming elements. For $m = 3 > n = 2$, we have the P_4 non-conforming finite element. That is, the new element is of full P_4 space, two more degrees of freedom locally than Wu-Xu's element [19].

2 Definition of Nonconforming Elements

Let a 2D polygonal domain be triangulated by a quasi-uniform triangular grid of size h, $\mathcal{T}_h$. Let $\mathcal{E}_h$ denote the set of edges of $\mathcal{T}_h$, and $\mathcal{E}_h(\Omega)$ denote the set of internal edges. Given $e = K_1 \cap K_2$, the jump and average of a piecewise function v across it