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PROPERTIES OF TENSOR COMPLEMENTARITY PROBLEM AND SOME CLASSES OF STRUCTURED TENSORS[*†]

Yisheng Song[‡]

(*School of Math. and Information Science, Henan Normal University, Xinxiang 453007, Henan, PR China*)

Liqun Qi

(*Dept. of Applied Math., The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, PR China*)

Abstract

This paper deals with the class of Q-tensors, that is, a Q-tensor is a real tensor $\mathcal{A}$ such that the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$:

$$\text{finding an } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{x} \geq \mathbf{0}, \mathbf{q}+\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \text{ and } \mathbf{x}^\top(\mathbf{q}+\mathcal{A}\mathbf{x}^{m-1}) = 0,$$

has a solution for each vector $\mathbf{q} \in \mathbb{R}^n$. Several subclasses of Q-tensors are given: P-tensors, R-tensors, strictly semi-positive tensors and semi-positive R_0-tensors. We prove that a nonnegative tensor is a Q-tensor if and only if all of its principal diagonal entries are positive, and so the equivalence of Q-tensor, R-tensors, strictly semi-positive tensors was showed if they are nonnegative tensors. We also show that a tensor is an R_0-tensor if and only if the tensor complementarity problem $(\mathbf{0}, \mathcal{A})$ has no non-zero vector solution, and a tensor is a R-tensor if and only if it is an R_0-tensor and the tensor complementarity problem $(\mathbf{e}, \mathcal{A})$ has no non-zero vector solution, where $\mathbf{e} = (1, 1 \cdots, 1)^\top$.

Keywords Q-tensor; R-tensor; R_0-tensor; strictly semi-positive; tensor complementarity problem

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1 Introduction

Throughout this paper, we use small letters $x, u, v, \alpha, \cdots$, for scalars, small bold

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[‡]Corresponding author. E-mail: songyisheng1@gmail.com

letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, for tensors. All the tensors discussed in this paper are real. Let $I_n := \{1, 2, \cdots, n\}$, $\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n)^\top; x_i \in \mathbb{R}, i \in I_n\}$, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n; x \geq \mathbf{0}\}$, $\mathbb{R}^n_- := \{\mathbf{x} \in \mathbb{R}^n; x \leq \mathbf{0}\}$, $\mathbb{R}^n_{++} := \{\mathbf{x} \in \mathbb{R}^n; x > \mathbf{0}\}$, $\mathbf{e} = (1, 1, \cdots, 1)^\top$, and $\mathbf{x}^{[m]} = (x_1^m, x_2^m, \cdots, x_n^m)^\top$ for $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top$, where $\mathbb{R}$ is the set of real numbers, $\mathbf{x}^\top$ is the transposition of a vector $\mathbf{x}$, and $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x} > \mathbf{0}$) means $x_i \geq 0$ ($x_i > 0$) for all $i \in I_n$.

Let $A = (a_{ij})$ be an $n \times n$ real matrix. A is said to be a **Q-matrix** iff the linear complementarity problem, denoted by $(\mathbf{q}, A)$,

$$\text{finding a } \mathbf{z} \in \mathbb{R}^n \text{ such that } \mathbf{z} \geq \mathbf{0}, \mathbf{q} + A\mathbf{z} \geq \mathbf{0}, \text{ and } \mathbf{z}^\top(\mathbf{q} + A\mathbf{z}) = 0, \tag{1.1}$$

has a solution for each vector $\mathbf{q} \in \mathbb{R}^n$. We say that A is a **P-matrix** iff for any nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$, there exists an $i \in I_n$ such that $x_i(Ax)_i > 0$. It is well-known that A is a P-matrix if and only if the linear complementarity problem $(\mathbf{q}, A)$ has a unique solution for all $\mathbf{q} \in \mathbb{R}^n$. Xiu and Zhang [1] also gave the necessary and sufficient conditions of P-matrices. A good review of P-matrices and Q-matrices can be found in the books by Berman and Plemmons [2], and Cottle, Pang and Stone [3].

Q-matrices and P(P_0)-matrices have a long history and wide applications in mathematical sciences. Pang [4] showed that each semi-monotone R_0-matrix is a Q-matrix. Pang [5] gave a class of Q-matrices which includes N-matrices and strictly semi-monotone matrices. Murty [6] showed that a nonnegative matrix is a Q-matrix if and only if all its diagonal entries are positive. Morris [7] presented two counterexamples of the Q-Matrix conjectures: a matrix is Q-matrix solely by considering the signs of its subdeterminants. Cottle [8] studied some properties of complete Q-matrices, a subclass of Q-matrices. Kojima and Saigal [9] studied the number of solutions to a class of linear complementarity problems. Gowda [10] proved that a symmetric semi-monotone matrix is a Q-matrix if and only if it is an R_0-matrix. Eaves [11] obtained the equivalent definition of strictly semi-monotone matrices, a main subclass of Q-matrices.

On the other hand, motivated by the discussion on positive definiteness of multivariate homogeneous polynomial forms [12-14], in 2005, Qi [15] introduced the concept of positive (semi-)definite symmetric tensors. In the same time, Qi also introduced eigenvalues, H-eigenvalues, E-eigenvalues and Z-eigenvalues for symmetric tensors. It was shown that an even order symmetric tensor is positive (semi-)definite if and only if all of its H-eigenvalues or Z-eigenvalues are positive (nonnegative) ([15, Theorem 5]). Various structured tensors have been studied well, such as, Zhang, Qi and Zhou [16] and Ding, Qi and Wei [17] for M-tensors, Song and Qi [18] for P-(P_0)tensors and B-(B_0)tensors, Qi and Song [19] for positive (semi-)definition of