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CONVERGENCE ANALYSIS OF NITSCHE EXTENDED FINITE ELEMENT METHODS FOR H(CURL)-ELLIPTIC INTERFACE PROBLEMS

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Abstract. An H(curl)-conforming Nitsche extended finite element method is proposed for H(curl)-elliptic interface problems in three dimensional Lipschitz domains with smooth interfaces. Under interface-unfitted meshes, the continuous problems are discretized by an H(curl)-conforming extended finite element space, which is constructed based on the the lowest order of second family Nédélec edge elements (Whitney elements). A stabilization term defined on transmission faces is added to the standard discrete bilinear form. Stability results and the optimal error estimate in the parameter-dependent H(curl)-norm are derived, which are both uniform with respect to not only the mesh size and the interface position but also the physical parameters. Numerical experiments are carried out to validate theoretical results.

Key words. Nitsche extended finite element method, H(curl)-elliptic interface problems, interface-unfitted meshes, the lowest order of second family Nédélec edge elements.

1. Introduction

A motivation for considering H(curl)-elliptic interface problems comes from the modeling of electromagnetic fields. In some electric machine applications, engineers need to solve an H(curl)-elliptic interface problem at each time step. Due to the large variety of applications in scientific computing and engineering, there have been a lot of work about the numerical approximations and convergence analyses for time-dependent Maxwell interface equations, stationary Maxwell interface equations and also other related models, such as [10], [13], [11], [33], [26], [28], [14], [30], [29], [19], [34], [3], [4], [22] and so on.

Among these papers, there are fitted-mesh methods ([28], [30], [29]), extended finite element methods with unfitted-meshes ([33]), adaptive immersed finite element methods with unfitted-meshes ([13]), Lagrange multiplier methods ([11], [3], [4])and so on. The optimal error estimates were obtained under interface-fitted meshes in [28], [30], [29]. Unfortunately, it is usually a time-consuming and nontrivial task to construct a good fitted-mesh for problems with moving interface or geometrically complicated interface. To avoid the expensive remeshing requirements, researchers pay more attention to unfitted-mesh methods. In this paper, we focus on one kind of interface-unfitted mesh methods−the extended finite element method.

The extended finite element method (XFEM) was first proposed by T. Belytschko and T. Black in [1] to deal with elastic equations in a cracked domain. In [23], A. Hansbo and P. Hansbo combined this method with Nitsche's method together, introduced a new method named Nitsche-XFEM. They successfully applied this new method to elliptic interface problems and obtained optimal error estimates independently of the interface position with respect to the mesh. Later, Nitsche-XFEM was taken to solve other elasticity and Stokes interface problems, such as

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[24], [31], [15], [32] and so on. As for the time-harmonic Maxwell equations, authors ([33]) study XFEM in two dimensional domains.

In this paper, we propose an H(curl)-conforming Nitsche extended finite element method for the H(curl)-elliptic interface problems in three dimensions. The extended finite element space is based on the lowest order of second family Nédélec edge elements. The discrete approximation scheme is formed by the standard bilinear formulation and a stabilization term defined on the transmission faces. By the help of the stabilization term, stable results and the optimal convergent order are derived, independent of not only the mesh size but also the interface position. Harmonic weights (see [41]) are applied in this paper, which make sure that all results are robust with respect to the physical parameters. In addition, comparing with the Lagrange multiplier method, we also have fewer degrees of freedom.

The layout of this paper is organized as follows. In Section 2, we define some notations, give the weak form of the original H(curl)-elliptic interface problem and construct its discrete formulation. Section 3 introduces some necessary assumptions and auxiliary lemmas. The stability properties containing the continuity and the coercivity are analyzed in Section 4. Section 5 shows the optimal error estimation under a parameter-dependent H(curl)-norm. Numerical experiments are presented in Section 6 to validate the theoretical results. Section 7 discusses the final conclusion.

Throughout this paper, we use bold typefaces to distinguish vectors from scalars, such as $\mathbf{E}$ and $\mathbf{H^2}(\Omega)$, denoting a vector function $\mathbf{E} = (E_1,\ E_2,\ E_3)$ and a vector space $\mathbf{H^2}(\Omega) = [H^2(\Omega)]^3$, respectively. $\mathbf{x} = (x_1,\ x_2,\ x_3) \in \mathbb{R}^3$ denotes the position of one point in the three dimensional space. Constants c or C with or without subscripts will be used to denote different positive constants which are independent of the mesh size, the physical parameters, and the interface location relative to the mesh.

2. H(curl)-elliptic interface problem

2.1. Weak formulation. Consider the following H(curl)-elliptic interface problem in the domain $\Omega \subseteq \mathbb{R}^3$

$$
\begin{aligned}
\mathbf{curl}(\alpha \mathbf{curl}\ \mathbf{u}) + \beta \mathbf{u} &= \mathbf{f} \text{ in } \Omega_1 \cup \Omega_2, \\
[\mathbf{n}_\Gamma \times \mathbf{u}] &= \mathbf{0} \text{ on } \Gamma, \\
[\mathbf{n}_\Gamma \times (\alpha \mathbf{curl}\ \mathbf{u})] &= \mathbf{0} \text{ on } \Gamma, \\
\mathbf{n} \times \mathbf{u} &= \mathbf{0} \text{ on } \partial\Omega,
\end{aligned} \tag{1}
$$

where Γ is a C^2-smooth boundary of a simple connected Lipschitz polyhedral domain Ω_1 with $\overline{\Omega}_1 \subseteq \Omega$ and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$, vectors $\mathbf{n}_\Gamma$, $\mathbf{n}$ represent the unit normal vector on Γ pointing from Ω_1 to Ω_2 and the unit outward normal vector of $\partial\Omega$ respectively, see Figure 1. For a suitable scalar function v, its jump across the interface is defined by $[v] = v|_{\Omega_1} - v|_{\Omega_2}$, and a component-wise application to a vector function. α, β are related physical parameters. For simplicity, we only concern about the case with β being a strictly positive constant and α being a piecewise constant in the domain Ω, namely

$$
\alpha = \begin{cases} \alpha_1 \text{ in } \Omega_1, \\ \alpha_2 \text{ in } \Omega_2. \end{cases}
$$