Journal of Nonlinear Modeling and Analysis
Volume 4, Number 1, March 2022, 103–113
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.103

# A Note on the Stefan-Boltzmann Problem for Heat Transfer in a Fin

Boris P. Belinskiy[1], John R. Graef[1,†] and Lingju Kong[1]

**Abstract** A fin is traditionally thought of as an extension of a surface to facilitate the transfer of heat away from a larger body to which it is attached. In this paper, the authors study some mathematical properties of a nonlinear heat transfer model for a fin and its relation to an associated linear model. Specifically, they prove that the solution exists and is unique, and they determine bounds for the temperature. Further, they prove the monotonicity of the temperature distribution, and they obtain an estimate for the maximal difference between the temperatures as determined by the nonlinear and linear models.

**Keywords** Heat transfer, Fin, Stefan-Boltzmann law, Existence and uniqueness, Dependence.

**MSC(2010)** 34B15.

## 1. Introduction

Extended surfaces, often called fins, are used in heat exchange devices to facilitate the transfer of the heat away from the main body. The usual physical assumptions in the heat transfer analysis of a fin are the following (see Lienhard IV and Lienhard V [15]):

(i) Heat transfer is 1-D.

(ii) Heat transfer is steady-state.

(iii) The conduction coefficient $k$, the convective heat transfer coefficient $h$, and the emmisivity $\epsilon$ are constant.

(iv) The temperature $T_b$ at the base of the fin is constant.

(v) The temperature $T_\infty$ of the fluid surrounding the fin is constant.

(vi) The body of the fin is a solid of revolution.

Then, it is the temperature variation along the fin that needs to be determined as a function of the distance from the base.

† the corresponding author.
Email address: Boris-Belinskiy@utc.edu (B. Belinskiy), John-Graef@utc.edu (J. R. Graef), Lingju-Kong@utc.edu (L. Kong)
[1] Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, Tennessee 37403, USA

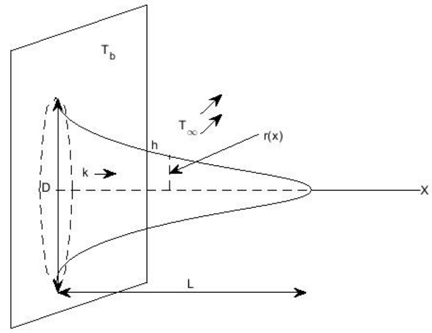


**Figure 1.** Profile of a fin

The geometry of the fin is as follows: First, it is located on the interval $[0, L]$ and is attached to a heated surface at $x = 0$. The radius of the cross section is given by $r(x) : [0, L] \to \mathbb{R}_+$, and the cross-sectional area, denoted by $A(x)$, is $A(x) = \pi r^2(x)$. The differential of surface area becomes $dA_s := P(x)dx = 2\pi r(x)\sqrt{1 + [r'(x)]^2}dx$.

We will denote the temperature distribution in the fin by $T(x) : [0, L] \to \mathbb{R}_+$. Then, the (steady-state) heat equation can be obtained from the energy balance for the region between $x$ and $x + dx$; it has the form (see [6, 15, 17, 19])

$$k\frac{d}{dx}\left(A\frac{dT}{dx}\right) - h\,\frac{dA_s}{dx}\,(T - T_\infty) - \epsilon\sigma\frac{dA_s}{dx}(T^4 - T_\infty^4) = 0, \quad x \in (0, L). \tag{1.1}$$

The third term on the left hand side of this equation represents the amount of heat transferred from the fin per unit area due to radiation. Here, $\epsilon$ is the emmisivity of the fin face, and $\sigma$ is the Stefan-Boltzmann constant. It is the presence of the radiation effect that makes the equation nonlinear. Since the constant $\sigma \approx 5.67 \times 10^{-8} W m^{-2} K^{-4}$ is small, at low temperatures the third term on the left hand side of (1.1) may be neglected so that the equation reduces to the linear one

$$k\frac{d}{dx}\left(A\frac{dT}{dx}\right) - h\,\frac{dA_s}{dx}\,(T - T_\infty) = 0, \quad x \in (0, L). \tag{1.2}$$

In addition to equations (1.1) and (1.2), we need to formulate the boundary conditions (BCs) that are to be satisfied.

The temperature at $x = 0$ is assumed to be the same as that of the base, which is namely $T_b$. At the right hand endpoint $x = L$, we assume that we have an adiabatic condition. Thus, our boundary conditions become

$$T(0) = T_b \quad \text{and} \quad \left(A\frac{d}{dx}T(x)\right)\bigg|_{x=L} = 0. \tag{1.3}$$

Boundary conditions other than (1.3) have been considered in the literature. For example, see [6, 15, 19].

**Remark 1.1.** In [17], the (steady-state) heat equation for a circular fin is given in the form (as adjusted to our notation)

$$\frac{d}{dr}\left(kA\frac{dT}{dr}\right) - h\,P\,(T - T_\infty) - \epsilon\sigma\,P\,(T^4 - T_\infty^4) = 0, \quad r \in (r_b, r_t), \tag{1.4}$$