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Exact Loop Wave Solutions and Cusp Wave Solutions of the Fujimoto-Watanabe Equation*

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Abstract In this paper, the theory of dynamical systems is employed to investigate loop waves and cusp waves of the Fujimoto-Watanabe equation. These waves contain solitary loop waves, periodic loop waves, peakons and periodic cusp waves. Under fixed parameter conditions, their exact explicit parametric expressions are given.

Keywords Fujimoto-Watanabe equation, Solitary loop wave, Periodic loop wave, Peakon, Periodic cusp wave.

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1. Introduction

The nonlinear equation

$$u_t - u^3 u_{xxx} - 3u^2(u_x u_{xx} + \alpha u_x) = 0 \tag{1.1}$$

was derived by Fujimoto and Watanabe [9], and is now known as the Fujimoto-Watanabe equation. Sakovich [18] showed that equation (1.1) can be connected with the famous KdV equation. Du [5] obtained some implicit expressions of traveling wave solutions of equation (1.1) by using an irrational equation method. Liu [14] gave the classifications of traveling wave solutions of equation (1.1) through the method of complete discrimination system. In [19], Shi and Wen obtained the implicit expressions of solitary wave solutions, periodic wave solutions, kink-like wave solutions and antikink-like wave solutions of equation (1.1). In [20–22], Shi and Wen continued to study three extension Fujimoto-Watanabe equations, and got some traveling wave solutions. These results enrich the research work of Fujimoto-Watanabe equation. The traveling wave of research on nonlinear differential equation has been greatly emphasized, some pieces of literature [1–3, 7, 10, 17] have shown some traveling wave solutions and properties of nonlinear equation. These results improve the content of traveling wave.

Here, our aim of this paper is to use the bifurcation method of planar systems and simulation method of differential equations [4, 6, 8, 11–13, 15, 16, 23–28] to investigate the loop wave solutions and cusp wave solutions of equation (1.1). The exact

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representation of loop wave solutions and cusp wave solutions of equation (1.1) are obtained. The planar graphs of the loop wave solutions and cusp wave solutions are shown under some parameters. These results are new.

For the given constant c, substituting $u(x,t) = \phi(\xi)$ with $\xi = x - ct$ in equation (1.1), it follows that

$$-c\phi' - \phi^3\phi''' - 3\phi^2(\phi'\phi'' + \alpha\phi') = 0. \tag{1.2}$$

Integrating equation (1.2) once with respect to ξ, we have the following traveling wave equation

$$\phi^3\phi'' = -\alpha\phi^3 - c\phi + g = 0, \tag{1.3}$$

where g is integral constant. Letting $\phi' = y$, then equation (1.3) becomes a planar system

$$\begin{cases} \dfrac{d\phi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{-\alpha\phi^3 - c\phi + g}{\phi^3}. \end{cases} \tag{1.4}$$

Clearly, on the straight line $\phi = 0$, system (1.4) is discontinuous. Such is called singular traveling wave system. Obviously, system (1.4) has the first integral

$$H(\phi, y) = \frac{y^2}{2} + \frac{2\alpha\phi^3 - 2c\phi + g}{2\phi^2} = h. \tag{1.5}$$

Let

$$f(\phi) = -2\alpha\phi^3 + 2h\phi^2 + 2c\phi - g. \tag{1.6}$$

According to the lemma of [19], we let the $(\phi^*, 0)$ is a saddle of system (1.4). We have main results as follows.

Proposition 1.1. *(1) If $\alpha < 0$, $c > 0$, and $g < 0$, or $\alpha < 0$, $c < 0$, and $g < 0$, then equation (1.1) has a solitary loop wave, and its parametric type is as follows:*

$$\begin{cases} \phi = \phi_1^* + (\phi^* - \phi_1^*)\tanh^2\left[\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w\right], \\ \xi = \sqrt{\frac{2}{\alpha(\phi_1^*-\phi^*)}}\left\{\phi^*\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w - (\phi^* - \phi_1^*)\tanh\left[\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w\right]\right\}, \end{cases} \tag{1.7}$$

where w is a parameter variable, $h = H(\phi^*, 0)$, ϕ_1^* and ϕ^* are a simple real zero and a double real zero of $f(\phi)$ and $\phi_1^* \le \phi < \phi^*$.

(2) If $\alpha > 0$, $c < 0$, and $g < 0$, or $\alpha > 0$, $c > 0$, and $g < 0$, then equation (1.1) has a solitary loop wave, and has parametric type as follows:

$$\begin{cases} \phi = \phi_1^* - (\phi_1^* - \phi^*)\tanh^2\left[\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w\right], \\ \xi = \sqrt{\frac{2}{\alpha(\phi_1^*-\phi^*)}}\left\{\phi^*\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w + (\phi_1^* - \phi^*)\tanh\left[\sqrt{\frac{\alpha(\phi_1^*-\phi^*)}{2}}w\right]\right\}, \end{cases} \tag{1.8}$$

where w is a parameter variable, $h = H(\phi^*, 0)$, ϕ_1^* and ϕ^* are a simple real zero and double real zero of $f(\phi)$, and $\phi^* < \phi \le \phi_1^*$.