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A Weighted Singular Trudinger-Moser Inequality

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Abstract. In this paper, we obtained the extremal function for a weighted singular Trudinger-Moser inequality by blow-up analysis in the Euclidean space $\mathbb{R}^2$. This extends recent results of Hou (J. Inequal. Appl., 2018) and similar result was proved by Zhu (Sci. China Math., 2021).

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1 Introduction

Let $\Omega\subset\mathbb{R}^2$ be a smooth bounded domain and $W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the norm $||u||_{W_0^{1,2}(\Omega)}=(\int_\Omega|\nabla u|^2\mathrm{d}x)^{1/2}$, where ∇ denotes the gradient operator on $\mathbb{R}^2$ and $||\cdot||_2$ is the standard L^2-norm. The classical Trudinger-Moser inequality [1–5] states the following:

$$\sup_{u\in W_0^{1,2}(\Omega),\int_\Omega|\nabla u|^2\mathrm{d}x=1}\int_\Omega e^{\alpha u^2}\mathrm{d}x<+\infty,\quad \forall\alpha\leq 4\pi, \tag{1.1}$$

where 4π is the best constant. The best means that when $\alpha>4\pi$, all integrals in (1.1) are finite, but the supremum is infinite. It is interesting to study whether the supremum in (1.1) can be attained or not. The first case for the attainability was proved by Carleson-Chang [1] when Ω is a ball in $\mathbb{R}^n$ $(n\geq 2)$. Then Struwe [6] extend the above result when Ω is close to a ball in measure and Flucher [7] get the relevant results with arbitrary domains in $\mathbb{R}^2$. Later, Lin [8] derived a result with bounded domain in $\mathbb{R}^n$ $(n\geq 2)$. This inequality (1.1) takes many forms. Chang - Yang [9] generalized (1.1) to the following form:

$$\sup_{u\in W^{1,2}(\Omega),\int_\Omega u\mathrm{d}x=0,\int_\Omega|\nabla u|^2\mathrm{d}x=1}\int_\Omega e^{2\pi u^2}\mathrm{d}x<+\infty. \tag{1.2}$$

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Recently, Hou [10] considers the classical Trudinger - Moser inequality (1.1) with weights, namely, for any $0 \le \alpha < \lambda_1(\Omega)$,

$$\sup_{u \in W_0^{1,2}(\Omega), ||u||_{1,\alpha} \le 1} \int_\Omega h e^{4\pi u^2} dx < +\infty, \tag{1.3}$$

where $||u||_{1,\alpha} = \left(\int_\Omega |\nabla u|^2 dx - \alpha \int_\Omega u^2 dx\right)^{1/2}$ and $\lambda_1(\Omega)$ is the first eigenvalue of the Laplace operator with respect to the Dirichlet boundary condition and positive function $h \in C^0(\overline{\Omega})$, with $h \not\equiv 0$.

Adimurthi and Sandeep [11] studied the singular version of (1.1). Assume $0 \in \Omega$, $\forall \beta \in (0,1)$, there holds

$$\sup_{u \in W_0^{1,2}(\Omega), \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^{2\beta}} dx < +\infty, \quad \forall \alpha < 4\pi(1-\beta), \tag{1.4}$$

where $4\pi(1-\beta)$ is the best constant. On this topic, we refer the reader to Adimurthi-Yang [12], Csato-Roy [13], Iula-Mancini [14], Li-Yang [15], Yang-Zhu [16,17]. Particularly, Zhu [18] generalized (1.4) to the following conclusion: $0 \in \partial\Omega$, for any $\beta \in (0,1)$, there have

$$\sup_{u \in W^{1,2}(\Omega), \int_\Omega u dx = 0, \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^{2\beta}} dx < +\infty, \quad \forall \alpha < 2\pi(1-\beta), \tag{1.5}$$

where $2\pi(1-\beta)$ is the best constant. There are many applications of (1.1) not only in Euclidean space, but also in Riemannian manifold, see for example, Ding-Jost-Li-Wang [19], Li [20,21], Li-Liu [22], Yang [23,24], Yang-Zhou [25] , Yang-Zhu [26] Zhu [27] and Yu [28].

In this paper, motivated by (1.3) and (1.5), we consider a weighted singular Trudinger-Moser inequality. Our conclusion reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain with $0 \in \partial\Omega$. For any $\beta \in (0,1)$, $g \in C^0(\overline{\Omega})$ with $g \ge 0$ and $g(0) > 0$, there holds*

$$\sup_{u \in W^{1,2}(\Omega), \int_\Omega u dx = 0, \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega g \frac{e^{\alpha u^2}}{|x|^{2\beta}} dx < +\infty, \ \forall \alpha \le 2\pi(1-\beta), \tag{1.6}$$

where $2\pi(1-\beta)$ is the best constant. Moreover, the supremum can be attained.

In order to prove Theorem 1.1, we follow the lines of [18] and mainly use the method of blow-up analysis. As a comparison with results of Zhu [18], we should mention that, there are more detailed analysis to do. Particularly, we use more coordinate transformations in our situation.

The remaining part of this article is organized as follows: In Section 2, we prove $2\pi(1-\beta)$ is the best possible constant by constructing test functions; In Section 3, we