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A High Order Bound Preserving Finite Difference Linear Scheme for Incompressible Flows

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Abstract. We propose a high order finite difference linear scheme combined with a high order bound preserving maximum-principle-preserving (MPP) flux limiter to solve the incompressible flow system. For such problem with highly oscillatory structure but not strong shocks, our approach seems to be less dissipative and much less costly than a WENO type scheme, and has high resolution due to a Hermite reconstruction. Spurious numerical oscillations can be controlled by the weak MPP flux limiter. Numerical tests are performed for the Vlasov-Poisson system, the 2D guiding-center model and the incompressible Euler system. The comparison between the linear and WENO type schemes, with and without the MPP flux limiter, will demonstrate the good performance of our proposed approach.

AMS subject classifications: 65M06, 65M20, 35M13

Key words: Finite difference scheme, high order Hermite reconstruction, MPP flux limiter, incompressible flow, Vlasov-Poisson.

1 Introduction

In this paper, we are interested in the numerical approximation of incompressible transport equations as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathbf{U}\cdot\nabla\rho = 0, \\ \operatorname{div}\mathbf{U} = 0, \end{cases} \tag{1.1}$$

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where $\mathbf{U}$ represents the advection field and ρ is a nonnegative density.

A typical example of application is the well known Vlasov-Poisson (VP) system arising in collisionless plasma physics. It describes the time evolution of particles under the effects of self-consistent electrostatic field and reads

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}f+\mathbf{E}\cdot\nabla_{\mathbf{v}}f=0, \tag{1.2}$$

$f:=f(t,\mathbf{x},\mathbf{v})$ is the distribution function in the phase space $(\mathbf{x},\mathbf{v})\in\mathbb{R}^d\times\mathbb{R}^d, d=1,2,3$. $\mathbf{E}:=\mathbf{E}(t,\mathbf{x})$ is the electric field, which can be determined by the Poisson's equation from an electric potential function $\Phi(t,\mathbf{x})$

$$\mathbf{E}(t,\mathbf{x})=-\nabla_{\mathbf{x}}\Phi(t,\mathbf{x}),\quad -\Delta_{\mathbf{x}}\Phi(t,\mathbf{x})=\rho(t,\mathbf{x}). \tag{1.3}$$

The charge density $\rho(t,\mathbf{x})$ is defined as

$$\rho(t,\mathbf{x})=\int_{\mathbb{R}^d}f(t,\mathbf{x},\mathbf{v})d\mathbf{v}.$$

Another example is the two dimensional guiding-center model, which describes the evolution of the charge density ρ in a highly magnetized plasma in the transverse plane of a tokomak, is given by

$$\begin{cases}\dfrac{\partial\rho}{\partial t}+\mathbf{U}\cdot\nabla\rho=0,\\ -\Delta\Phi=\rho,\end{cases} \tag{1.4}$$

where $\mathbf{U}=(-\Phi_y,\Phi_x)$ is a divergence free velocity. The two dimensional guiding center model can also be referred as an asymptotic model of the VP system by averaging in the velocity phase space, for details, see [28]. We notice that the guiding-center model (1.4) is in the same form as the two dimensional incompressible Euler equations in the vorticity stream function formulation, which describes the evolution of vortices in fluid hydrodynamics.

For the models mentioned above, they all have a transport equation coupled with a Poisson's equation for the advection velocity, and moreover, the advection velocity is divergence free. In the following, we refer them as incompressible flow models.

Many numerical schemes have been proposed for solving these models, especially recently, high order schemes are very attractive due to their high resolutions for such problems with rich solution structures. For example, deterministic methods, there are finite difference, finite volume and finite element Eulerian methods [10,12,18–20,43,51,53, 54,56,59], semi-Lagrangian methods [4,5,7,8,13,14,23,31–35,38,45,46,52,58], and discontinuous Galerkin finite element methods [3,6,11,15,22,24,29,55,60], also see many other references therein. However, due to the highly oscillatory structure of such problems, linear type schemes for these problems would show significant spurious numerical oscillations, which might get worse with increased orders. Weighted essentially nonoscillotry (WENO) reconstruction, which was originally developed in the presence of both shocks