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Monge-Ampère Equation with Bounded Periodic Data

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Abstract. We consider the Monge-Ampère equation $\det(D^2u) = f$ in $\mathbb{R}^n$, where f is a positive bounded periodic function. We prove that u must be the sum of a quadratic polynomial and a periodic function. For $f \equiv 1$, this is the classic result by Jörgens, Calabi and Pogorelov. For $f \in C^\alpha$, this was proved by Caffarelli and the first named author.

Key Words: Monge-Ampère equation, Liouville theorem.

AMS Subject Classifications: 53C20, 53C21, 58J05, 35J60

1 Introduction

A classic theorem of Jörgens [17], Calabi [11] and Pogorelov [20] states that any classical convex solution of

$$\det(D^2u) = 1 \quad \text{in } \mathbb{R}^n$$

must be a quadratic polynomial.

A simpler and more analytical proof, along the lines of affine geometry, was later given by Cheng and Yau [12]. The theorem was extended by Caffarelli [1] to viscosity solutions. Another proof of the theorem was given by Jost and Xin [18]. Trudinger and Wang [21] proved that if Ω is an open convex subset of $\mathbb{R}^n$ and u is a convex C^2 solution of $\det(D^2u) = 1$ in Ω with $\lim_{x\to\partial\Omega} u(x) = \infty$, then $\Omega = \mathbb{R}^n$. Ferrer, Martínez and Milán [14, 15] extended the above Liouville type theorem in dimension two. Caffarelli and the first named author [8,9] made two extensions, and one of them includes periodic data.

More specificly, assume for some $a_1, \cdots, a_n > 0$, f satisfies

$$f(x + a_ie_i) = f(x), \quad \forall x \in \mathbb{R}^n, \quad 1 \le i \le n, \tag{1.1}$$

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where $e_1 = (1,0,\cdots,0),\cdots,e_n = (0,\cdots,0,1)$.

Consider the Monge-Ampère equation

$$\det(D^2u) = f \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Theorem A ([9]). *Let $f \in C^\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$ with $f > 0$ satisfy* (1.1)*, and let $u \in C^2(\mathbb{R}^n)$ be a convex solution of* (1.2)*. Then there exist $b \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A with*

$$\det A = \fint_{\prod_{1\le i\le n}[0,a_i]} f,$$

such that

$$v := u - \frac{1}{2}x^TAx - b\cdot x$$

is a_i-periodic in i-th variable, i.e.,

$$v(x+a_ie_i) = v(x), \quad \forall x \in \mathbb{R}^n, \quad 1 \le i \le n.$$

For applications, it is desirable to study the problem with less regularity assumption on f. It was conjectured in [9], see Remark 0.5 there, that Theorem A remains valid for $f \in L^\infty(\mathbb{R}^n)$ satisfying

$$0 < \inf_{\mathbb{R}^n} f \le \sup_{\mathbb{R}^n} f < \infty.$$

We confirm the conjecture in Theorem 1.2 below.

We first recall the definition of a solution of (1.2) in the Alexandrov sense.

Let u be a convex function in an open set Ω of $\mathbb{R}^n$. For $y \in \Omega$, denote

$$\nabla u(y) = \{p \in \mathbb{R}^n | u(x) \ge u(y) + p\cdot(x-y),\ \forall x \in \Omega\}$$

the generalized gradient of u at y.

For $f \in L^\infty(\Omega)$ with $f \ge 0$ a.e., u is called a solution of

$$\det(D^2u) = f \quad \text{in } \Omega$$

in the Alexandrov sense if u is a convex function in Ω and $|\nabla u(O)| = \int_O f$, for every open set $O \subset \Omega$.

Similarly, for a symmetric $n \times n$ matrix A, we say that $v \in C^{0,1}(\Omega)$ is a solution

$$\det(A + D^2v) = f \quad \text{in } \Omega$$

in the Alexandrov sense if $u := \frac{1}{2}x^TAx + v$ is convex in Ω and satisfies

$$\det(D^2u) = f \quad \text{in } \Omega$$

in the Alexandrov sense.

Our first result is the existence and uniqueness of periodic solutions for $f \in L^\infty$.