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LOW ORDER MIXED FINITE ELEMENT APPROXIMATIONS OF THE MONGE-AMPÈRE EQUATION

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Abstract. In this paper, we are interested in the analysis of the convergence of a low order mixed finite element method for the Monge-Ampère equation. The unknowns in the formulation are the scalar variable and the discrete Hessian. The distinguished feature of the method is that the unknowns are discretized using only piecewise linear functions. A superconvergent gradient recovery technique is first applied to the scalar variable, then a piecewise gradient is taken, the projection of which gives the discrete Hessian matrix. For the analysis we make a discrete elliptic regularity assumption, supported by numerical experiments, for the discretization based on gradient recovery of an equation in non divergence form. A numerical example which confirms the theoretical results is presented.

Key words. Monge-Ampère, mixed finite element, gradient recovery, non divergence form.

1. Introduction

In this paper, we analyze a linear finite element discretization of the elliptic Monge-Ampère equation for smooth solutions on a convex polygonal domain. The method is a variant of the method introduced in [15] for which numerical experiments for both smooth and non-smooth solutions were reported in [20]. Let Ω be a convex polygonal domain in $\mathbb{R}^2$ endowed with a triangulation $\mathcal{T}_h$ which is conforming and quasi-uniform. For the purpose of our analysis, we further assume the triangulation to be uniform, i.e. two triangles sharing an edge form a parallelogram. Let V_h denote the space of piecewise linear continuous functions on Ω and let Σ_h denote the space of piecewise linear continuous 2×2 matrix fields on Ω. Our goal is to seek an element $u_h \in V_h$ which approximates the unique strictly convex $C^4(\overline{\Omega})$ solution u (when it exists) of the problem

$$\begin{aligned} \det(D^2 u) &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

The right hand side function $f \in C^2(\overline{\Omega})$ is assumed to satisfy $f > 0$. The boundary function $g \in C(\partial\Omega)$ is also given and assumed to extend to a $C^4(\overline{\Omega})$ convex function. Here we use $\det(D^2u)$ to denote the determinant of the Hessian matrix $D^2u = \big(\partial^2 u/(\partial x_i \partial x_j)\big)_{i,j=1,2}$.

The discrete problem is to find $u_h \in V_h$ such that $u_h = g_h$ on $\partial\Omega$ and

$$\int_\Omega (f - \det H(u_h)) v \, dx = 0, \forall v \in V_h \cap H_0^1(\Omega), \tag{2}$$

where $H(u_h)$, the discrete Hessian of u_h, is an element of Σ_h defined by

$$\int_\Omega H(u_h) : \mu \, dx = \int_\Omega (DG_h u_h) : \mu \, dx, \forall \mu \in \Sigma_h. \tag{3}$$

The operator $G_h : V_h \to V_h \times V_h$ in (3) is taken as the weighted average gradient recovery operator and is somehow a substitution for the gradient operator. The

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finite element function g_h is the standard finite element interpolation of the continuous function g in V_h. For two matrices A and B, $A : B$ denotes their Frobenius inner product. We denote by $\mathcal{E}_h^i$ the set of interior edges of $\mathcal{T}_h$and by $\mathcal{N}_h$ the set of vertices of $\mathcal{T}_h$. For a vector field v, Dv denotes its piecewise gradient vector, the matrix field with rows the gradients of the corresponding components of v,.

The Monge-Ampère operator appears in a number of problems where the solution is known to be smooth. For example, it appears in the study of von Kármán model for plate buckling [5]. It is argued in [17] that for meteorological applications for which legacy finite element codes are used for the discretization of other differential operators, it could be advantageous to use a finite element discretization as well for the Monge-Ampère operator. The readers are referred to [9, 21] and the references therein for a review of numerical methods for Monge-Ampère type equations.

Problem (2) with the discrete Hessian (3) is equivalent to the following mixed formulation: find $(u_h, \sigma_h) \in V_h \times \Sigma_h$ such that $u_h = g_h$ on $\partial\Omega$

$$\begin{aligned} \int_\Omega (f - \det \sigma_h) v \, dx &= 0, \quad \forall v \in V_h \cap H_0^1(\Omega) \\ \int_\Omega \sigma_h : \mu \, dx &= \int_\Omega (DG_h u_h) : \mu \, dx, \quad \forall \mu \in \Sigma_h. \end{aligned} \tag{4}$$

Analysis of discretizations similar to (2) and (4) for cubic and higher order elements were conducted in [20, 3]. Problem (2), with the gradient recovery operator replaced by the piecewise gradient in a weak formulation of (3), was proposed in [15, 20] for quadratic and higher order approximations, c.f. Remark 3.2 below. See also [20] for a version with linear approximations. Related ideas can be found in [14, 10, 16, 11]. Our error analysis is based on the above formulation (4). We use the same argument as in [20, 3].

In addition, we make a discrete elliptic regularity assumption for the discretization based on a gradient recovery operator of the non divergence form of a linear elliptic equation. We support this assumption with numerical experiments. The linear elliptic equation considered is the linearization of the Monge-Ampère equation and can be written in both divergence and non divergence forms. A discrete elliptic regularity approach for a linear equation in divergence form was first used in [19] for interior penalty methods for the Monge-Ampère equation on a smooth domain. It was recently used in [2] for a mixed method under an assumption of elliptic regularity for the linearization of the continuous problem.

We show that the piecewise gradient of the recovered gradient of the finite element solution converges at a rate $\mathcal{O}(h)$ to the piecewise gradient of the recovered gradient of the interpolant in the L^p norm with $|\ln h| \leq p \leq 2|\ln h|$, and the discrete Hessian converges at a rate $\mathcal{O}(h)$ in the L^∞ norm.

Our analysis is limited to uniform partitions of a convex polygonal domain so that we can take advantage of a superconvergent approximation property for the gradient recovery operator proved in [23], c.f. (11) below. We want to emphasize that although we only give the analysis on uniform meshes, numerical results indicate that the results may hold on general Delaunay triangulations. Elements of Σ_h can be required to be symmetric matrix fields to reduce the number of unknowns. The analysis of this paper also holds in that case.

The rest of the paper is organized as follows. In Section 2, we present some additional notation and preliminaries. In Section 3, we conduct the error estimate for the discrete Monge-Ampère equation. In section 4, we give numerical results for a smooth solution to support our theoretical results. Some conclusions are drawn in