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A Survey of Optimal Control Problems Evolved on Riemannian Manifolds

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Abstract. In this paper, we present our optimality results on optimal control problems for ordinary differential equations on Riemannian manifolds. For the problems with free states at the terminal time, we obtain the first and second-order necessary conditions, dynamical programming principle, and their relations. Then, we consider the problems with the initial and final states satisfying some inequality-type and equality-type constraints, and establish the corresponding first and second-order necessary conditions of optimal pairs in the sense of either spike or convex variations. For each of the above results concerning second-order optimality conditions, the curvature tensor of the underlying manifold plays a crucial role.

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Key words: Optimal control, necessary condition, dynamical programming principle, Riemannian manifold, curvature tensor.

1 Introduction

In this paper, we shall consider a general control system with its state constrained to a manifold (or roughly speaking, a curved space), and introduce our recent results on optimal control problems for such a system. There are several backgrounds that motivate us to study this sort of optimal control problems. First, when formulating some control problems in practice, the systems under consideration are actually evolved on some manifolds. For example, the control problem of the rotations of a rigid body is described by a system, with the state constrained to the set of all 3×3 orthogonal matrices with determinant 1 (see [2, Sect. 22.2, p. 358]), which is clearly not a linear space. As another

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example, in the control system modelling the Dubins car, the state is restricted to $\mathbb{R}^2\times S^1$, where S^1 is the unit circle in $\mathbb{R}^2$ (see [2, Sect. 13.5, p. 200]). Second, because the earth's surface is curved, many control systems actually have a hidden constraint: their states are constrained to manifolds. Finally, in view of the general relativity theory, both time and space are intrinsically curved, and therefore the control systems in flat spaces (i.e., the classical Euclidean spaces) are only ideal and special cases. Thus, either to solve concrete optimal control problems from practice, or to obtain more precise results theoretically, it is indispensable to study optimal control problems on manifolds.

We need some notations to be used later. Let $n\in\mathbb{N}$ and M be an n-dimensional, simply connected manifold. For each $x\in M$, we denote by T_xM, T_x^*M and $\mathcal{T}_s^r(x)$ $(r,s\in\mathbb{N}\cup\{0\})$ the tangent space, the cotangent space, and the set of all tensors of type (r,s) of M at x, respectively. Also, denote by $TM\equiv\{(x,X);x\in M,X\in T_xM\}$, $T^*M\equiv\{(x,\eta);x\in M,\eta\in T_x^*M\}$, $\mathcal{X}(M)$, $C^\infty(M)$ and $\mathcal{T}_s^r(M)$ the tangent bundle, the cotangent bundle, the set of smooth vector fields, the set of smooth functions on M and the set of all tensor fields of type (r,s) over M respectively. For $h\in C^\infty(M)$, we denote by dh the differential of h.

For a given $T>0$, a control set U, and a vector field f depending on the time $t\in[0,T]$ and the control $u\in U$, we consider the control system:

$$\dot{y}(t)=f(t,y(t),u(t)),\quad y(t)\in M,\quad a.e.\ t\in[0,T], \tag{1.1}$$

where $\dot{y}(t)=\frac{d}{dt}y(t)$ for $t\in[0,T]$, and the control function $u(\cdot)$ belongs to the following set

$$\mathcal{U}=\{u\colon [0,T]\to U;\ u(\cdot) \text{ is measurable}\}.$$

The main concern of this paper is about the optimality conditions on optimal control problems for the system (1.1).

First of all, for the control system (1.1), we consider a relatively simple optimal control problem:

(P_f) Given maps $f^0\colon \mathbb{R}\times M\times U\to\mathbb{R}$ and $h\colon M\to\mathbb{R}$ and a point $y_0\in M$, find a control $\bar{u}(\cdot)\in\mathcal{U}$, called an optimal control, which minimizes the functional:

$$J_f(u(\cdot))=\int_0^T f^0(t,y(t),u(t))dt+h(y(T)) \tag{1.2}$$

over the pairs $(u(\cdot),y(\cdot))$ satisfying (1.1), $y(0)=y_0$ and $u(\cdot)\in\mathcal{U}$. The corresponding trajectory $\bar{y}(\cdot)$ is called an optimal trajectory, and $(\bar{u}(\cdot),\bar{y}(\cdot))$ is called an optimal pair.

When M is a Euclidean space, the results on optimality conditions for Problem (P_f) are very rich. For examples, Lou [20] studies the second-order necessary conditions, Zhou [32] finds the relation between Pontryagin's maximum principle and dynamical programming principle, Frankowska and Hoehener [14] further discover the relation between the first and second-order necessary conditions and dynamical programming