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On Local Wellposedness of the Schrödinger-Boussinesq System

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Abstract. In this paper we prove that the Schrödinger-Boussinesq system with solution $(u,v,\ (-\partial_{xx})^{-\frac{1}{2}}v_t)$ is locally wellposed in $H^s\times H^s\times H^{s-1}$, $s\geqslant -1/4$. The local wellposedness is obtained by the transformation from the problem into a nonlinear Schrödinger type equation system and the contraction mapping theorem in a suitably modified Bourgain type space inspired by the work of Kishimoto, Tsugawa. This result improves the known local wellposedness in $H^s\times H^s\times H^{s-1}$, $s>-1/4$ given by Farah.

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1 Introduction

We study the Cauchy problem of the Schrödinger-Boussinesq system(SBq) in $(t,x)\in\mathbb{R}\times\mathbb{R}$:

$$i\partial_t u+\partial_x^2 u=uv, \tag{1.1}$$
$$\partial_t^2 v-\partial_x^2 v+\partial_x^4 v+\partial_x^2 v^2=\partial_x^2|u|^2, \tag{1.2}$$
$$u(0,x)=u_0(x), \tag{1.3}$$

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$$v(0,x)=v_0(x), \quad v_t(0,x)=v_1(x), \tag{1.4}$$

where function u is complex-valued, while v can be complex-valued or real-valued.

The system above appears in the study of interaction of solitons in optics and can be considered as a model describing interactions between short and intermediate long waves, see Makhankov [1], Yajima et al. [2], Nishikawa et al. [3]. The short wave term $u(t,x)$ is described by the Schrödinger equation (1.1) with the potential $v(t,x)$ satisfying the Boussinesq equation (1.2) and representing the intermediate long wave. When $\partial_x^2 v^2$ in (1.2) is replaced with $\partial_x^2\left(\beta|v|^{p-1}v\right)$, where $\beta\in\mathbb{R}$ is a constant, the (1.1)-(1.4) is called generalized Schrödinger-Boussinesq system and is studied in Linares and Navas [4] and so on. The results about wellposedness, numerical results, stability of solitary waves etc. for the generalized Schrödinger-Boussinesq system can be found in [5–8] and the references therein.

As for the local wellposedness, Linares and Navas [4] obtained that the solution (u,v) of the generalized Schrödinger-Boussinesq system is locally wellposed in $C([-T,T]; L^2(\mathbb{R}))\cap L^4([-T,T];L^\infty(\mathbb{R}))$. The main method of [4] is L^p-L^q estimates, more precisely, Linares and Navas derived the results by using the global smoothing effects established in [9] for the generalized Boussinesq equation and the standard Strichartz estimates for the Schrödinger equation. We remark that the results of [4] is suit for the Schrödinger-Boussinesq system as well, meanwhile from the method used in [4], it is not possible to obtain corresponding local wellposedness results of the generalized Schrödinger-Boussinesq system in higher dimensions, as the smoothing effects in [9] are the special case of Strichartz type estimates for the generalized Boussinesq equation and do not fit with the Sobolev embedding inequalities in higher dimensions.

Farah [10] has researched following Schrödinger-Boussinesq system

$$\begin{cases} iu_t+\partial_{xx}u=vu,\\ \partial_t^2 v-\partial_x^2 v+\partial_x^4 v=\partial_x^2|u|^2,\\ u(0,x)=u_0(x),\ \ v(0,x)=v_0(x),\ \ v_t(0,x)=(v_1)_x(x) \end{cases} \tag{1.5}$$

and used the classic Bourgain space method to obtain low regularity local wellposedness of $(u,v,\ (\partial_x)^{-1}v_t)$ in $H^s(\mathbb{R})\times H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$, $s>-1/4$, which can be applied to (1.1)-(1.4) as well. See also Farah, Pastor [11] for the periodic case. Farah [10] introduced norms

$$\|u\|_{X^{s,b}}=\left\|\left\langle\tau+\xi^2\right\rangle^b\langle\xi\rangle^s\widetilde{u}\right\|_{L^2_{\xi,\tau}} \quad \text{for the Schrödinger equation,} \tag{1.6}$$

$$\|v\|_{X_2^{s,b}}=\|\langle|\tau|-\gamma(\xi)\ \rangle^b\langle\xi\rangle^s\widetilde{v}\Big\|_{L^2_{\xi,\tau}},\ \ \gamma(\xi)=\sqrt{\xi^2+\xi^4}\ \ \text{for the Boussinesq equation,} \tag{1.7}$$

and applied the property

$$\frac{1}{c}\leqslant\frac{1+|x-y|}{1+\left|x-\sqrt{y^2+y}\right|}\leqslant c, \qquad \text{for } y\geqslant 0 \text{ and some constant } c>0 \tag{1.8}$$