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Intrinsic Formulation of the Kirchhoff-Love Theory of Nonlinearly Elastic Shallow Shells

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Abstract. The classical formulation of the Kirchhoff-Love theory of nonlinearly elastic shallow shells consists of a system of nonlinear partial differential equations and boundary conditions whose unknowns are the Cartesian components of the displacement field of the middle surface of the shell subjected to applied forces. We show that this system is equivalent to a system whose sole unknowns are the bending moments and stress resultants inside the middle surface of the shell. This system thus provides a direct method for computing the stresses appearing in such a shell, without any recourse to the displacement field. To this end, we first establish specific compatibility conditions of Saint-Venant type for the bending moments and stress resultants; we then identify the boundary conditions that these fields must satisfy.

AMS subject classifications: 74K25, 74B20, 53A05, 35J66

Key words: Nonlinearly elastic shallow shells, displacement-traction problem, Kirchhoff-Love theory, Saint-Venant compatibility conditions, Euler-Lagrange equation, intrinsic formulation.

1 Introduction

A shallow shell is a thin shell whose middle surface is "almost planar", in the sense that the principal curvatures of the middle surface of the shell are of the order of its thickness (the precise definition is given in Section 2).

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If a shallow shell is made of an elastic material and is subjected to external forces, the shell will undergo a deformation to reach an equilibrium state, where the corresponding internal stresses are given in terms of the displacement field and its partial derivatives by means of the constitutive equation of the elastic material from which the shell is made. The displacement field then satisfies a specific boundary value problem formed by a system of partial differential equations and boundary conditions defined over a three dimensional domain representing the reference configuration of the shell.

The classical Kirchhoff-Love theory for nonlinearly elastic shallow shells provides a way to compute the internal stresses and the displacement field inside a shallow shell by solving a boundary value problem defined over a two-dimensional domain, whose unknown is the vector field formed by the Cartesian components of the displacement field of the middle surface of the reference configuration of the shell.

More specifically, the classical formulation of the Kirchhoff-Love theory computes the internal stresses in the deformed shallow shell in two stages: First, the displacement field of the middle surface of the shallow shell is computed by solving a specific boundary value problem (Section 2); Second, the internal stresses are computed in terms of this displacement field by using the constitutive equation of the elastic material constituting the shell.

The objective of this paper is to provide a simpler way to compute the internal stresses in the deformed shallow shell by means of an intrinsic formulation, the main feature of which is to entirely eliminate the need of computing the displacement field. This is done by introducing a new boundary value problem whose sole unknowns are the two-dimensional stresses, or equivalently the strains, of the middle surface of the shell and by proving that the two-dimensional stresses found in this way coincide with those found by solving the classical boundary value problem of Kirchhoff-Love (Theorems 4.1 and 4.2).

More specifically, we show that the bending moments and stress resultants of the middle surface of the deformed shell are the symmetric tensor fields (all the notation used in this introduction is defined in Section 2)

$$(M_{\alpha\beta}):\overline{\omega}\to\mathbb{S}^2,\quad (N_{\alpha\beta}):\overline{\omega}\to\mathbb{S}^2$$

that satisfy the following boundary value problem (see Theorem 4.1):

$$\begin{aligned}
&-\partial_\beta N_{\alpha\beta}=p_\alpha && \text{in } \omega,\\
&-\partial_{\alpha\beta}M_{\alpha\beta}-\partial_\alpha\big(N_{\alpha\beta}[\partial_\beta\zeta_3+\varepsilon\partial_\beta h]\big)=p_3+\partial_\alpha q_\alpha && \text{in } \omega,\\
&\ N_{\alpha\beta}\nu_\beta=M_{\alpha\beta}\nu_\alpha\nu_\beta=0 && \text{on } \gamma_1,\\
&\ N_{\alpha\beta}\nu_\alpha(\partial_\beta\zeta_3+\varepsilon\partial_\beta h)+(\partial_\alpha M_{\alpha\beta})\nu_\beta+\partial_\tau(M_{\alpha\beta}\nu_\alpha\tau_\beta)=-q_\alpha\nu_\alpha && \text{on } \gamma_1,
\end{aligned}$$