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Fourth-Order Structure-Preserving Method for the Conservative Allen-Cahn Equation

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Abstract. We propose a class of up to fourth-order maximum-principle-preserving and mass-conserving schemes for the conservative Allen-Cahn equation equipped with a non-local Lagrange multiplier. Based on the second-order finite-difference semi-discretization in the spatial direction, the integrating factor Runge-Kutta schemes are applied in the temporal direction. Theoretical analysis indicates that the proposed schemes conserve mass and preserve the maximum principle under reasonable time step-size restriction, which is independent of the space step size. Finally, the theoretical analysis is verified by several numerical examples.

AMS subject classifications: 65N06, 65N12

Key words: Maximum-principle-preserving, mass-conserving scheme, the conservative Allen-Cahn equation.

1 Introduction

The classical Allen-Cahn (AC) equation was proposed by Allen and Cahn [1] in 1979 to describe the phenomenological model of the inverse phase boundary motion in crystals. As an important class of phase field models, the AC equation has been widely applied in image processing [2], mean curvature motion, materials science [3, 4], and so on. In recent years, many studies have been conducted on the classical AC equation [5–8].

The classical AC equation is considered as a well-known prototypical gradient flow

$$\partial_t u(x,t)=\epsilon^2\Delta u(x,t)+f(u(x,t)), \quad x\in\Omega, \quad t>0, \tag{1.1}$$

where $\Omega=[a,b]\subseteq\mathbb{R}$ is the bounded domain. The parameter $\epsilon>0$ and u usually represent the interfacial width and the difference between the concentrations of two mixtures' components, respectively. The symbol Δ denotes the usual Laplacian operator and $f(u)$ is the

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negative derivative of a polynomial double-well potential, i.e., $f(u)=-F'(u)$. Consider the initial and periodic boundary conditions

$$u(x,0)=u_0(x), \qquad x\in\Omega, \tag{1.2a}$$
$$u(a,t)=u(b,t), \qquad t\geq 0. \tag{1.2b}$$

The L^2 inner product and norm are denoted as

$$\langle f,g\rangle=\int_\Omega fg dx, \qquad \|f\|=\Big(\int_\Omega |f|^2 dx\Big)^{\frac{1}{2}},$$

respectively. The L^∞ norm is defined as

$$\|f\|_{L^\infty}=\max_{x\in\Omega}|f(x)|.$$

The energy functional of the classical AC equation is defined as

$$E[u]=\frac{\epsilon^2}{2}\langle\nabla u,\nabla u\rangle+\langle F(u),1\rangle=\int_\Omega\Big(\frac{\epsilon^2}{2}|\nabla u(x,t)|^2+F(u(x,t))\Big)dx, \tag{1.3}$$

where

$$F(u)=\frac{1}{4}(u^2-1)^2, \quad f(u)=-F'(u)=u-u^3.$$

By taking the L^2 inner product of Eq. (1.1) with $\partial_t u(x,t)$, we obtain

$$\frac{d}{dt}E[u(x,t)]=-\int_\Omega|\partial_t u(x,t)|^2dx\leq 0, \quad \forall t>0. \tag{1.4}$$

Thus, the classical AC equation satisfies the energy dissipation law. By taking the L^2 inner product of Eq. (1.1) with 1, we have

$$\frac{d}{dt}\int_\Omega u(x,t)dx=\epsilon^2\int_\Omega\Delta u(x,t)dx+\int_\Omega f(u(x,t))dx, \quad \forall t>0. \tag{1.5}$$

It can be proven that the classical AC equation can not conserve the mass unless

$$\int_\Omega f(u)dx=0.$$

In this paper, by introducing a Lagrange multiplier

$$\lambda=\frac{1}{|\Omega|}\int_\Omega f(u(x,t))dx,$$

the conservative modification of the classical AC equation is expressed as [30]

$$\partial_t u(x,t)=\epsilon^2\Delta u(x,t)+\bar{f}(u(x,t)), \quad x\in\Omega, \quad t>0, \tag{1.6}$$