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Three Indication Variables and Their Performance for the Troubled-Cell Indicator using K-Means Clustering

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Abstract. In Zhu, Wang and Gao (SIAM J. Sci. Comput., 43 (2021), pp. A3009–A3031), we proposed a new framework of troubled-cell indicator (TCI) using K-means clustering and the numerical results demonstrate that it can detect the troubled cells accurately using the KXRCF indication variable. The main advantage of this TCI framework is its great potential of extensibility. In this follow-up work, we introduce three more indication variables, i.e., the TVB, Fu-Shu and cell-boundary jump indication variables, and show their good performance by numerical tests to demonstrate that the TCI framework offers great flexibility in the choice of indication variables. We also compare the three indication variables with the KXRCF one, and the numerical results favor the KXRCF and the cell-boundary jump indication variables.

AMS subject classifications: 65M60, 35L60, 35L65, 35L67

Key words: Troubled-cell indicator, indication variable, discontinuous Galerkin method, shock detection, K-means clustering.

1 Introduction

The discontinuous Galerkin (DG) method is a stable, conservative and high-order accurate finite element method. Due to the discontinuous nature of the solution and test function space, it has the advantages of easy $h-p$ adaptations and boundary treatments and high efficiency in parallel implementation. The DG method has found popularity in

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a wide range of fields such as computational fluid dynamics [6], Hamilton-Jacobi equations [10], high-order time-dependent PDEs [27], elliptic problems [1] and many others.

The DG method we discuss in this paper is the Runge-Kutta (RK) DG method developed by Cockburn et al. in a series of papers [2–5]. It was intended for hyperbolic conservation laws whose solutions can develop discontinuities no matter how smooth the initial conditions are. Discontinuities can cause spurious oscillations in the numerical solution or even a failure of the numerical scheme. The RKDG method relies on a nonlinear limiter to ensure stability. A limiter usually first finds the locations of the discontinuities by marking the cells near the discontinuities as troubled cells, and then reconstructs the numerical solution in the detected cells so that spurious oscillations can be controlled and stability is achieved. The former part, named troubled-cell indicator (TCI) [17], is crucial to the RKDG method. If less troubled cells than necessary are detected, oscillations may appear; on the contrary, if too many troubled cells are detected, extra computational resources will be wasted.

TCIs can come from limiters and shock detection methods. There have been massive works on limiters/TCIs in the literature. An early work of a comparison of different TCIs was given by Qiu and Shu [16] in 2005. In their work, seven different TCIs, coupled with a weighted essentially non-oscillatory (WENO) solution reconstruction method [17], were compared for the RKDG method on uniform meshes. These TCIs were later investigated and compared for h-adaptive RKDG methods [31, 32]. A systematic review of limiters before 2014 is available in [7]. TCIs have been receiving attention in recent years. An automatic parameter selection strategy was proposed in [24], addressing the issue of problem-dependent parameters in TCIs. A modification of this work was later introduced by Gao et al. [9]. Fu and Shu [8] designed a simple and compact TCI that has just one parameter depending only on the DG polynomial degree. Zhu et al. [30] later extended their work to h-adaptive meshes. New techniques such as neural networks in deep learning have been employed to construct new type of TCIs, for example, the works by Ray and Hesthaven [18, 19] and by Sun et al. [22].

Many TCIs are originally designed for uniform (usually rectangular in two dimensions) meshes. Some of them can work for unstructured meshes but only few of them can be directly applied to irregular meshes (such as two-dimensional h-adaptive meshes where hanging nodes appear). In [33], we developed a new framework of TCIs that can be extended to unstructured meshes and irregular meshes. It was motivated by the work of Vuik and Ryan [24].

For any troubled-cell indication variable, the indication values usually display a sudden increase or decrease at a discontinuity, with respect to the neighboring values. Based on this knowledge, Vuik and Ryan reduced the troubled-cell indication to the outlier detection of the indication values from a local region. The Tukey's boxplot approach [23], which is commonly used in statistical analysis, is their choice to detect the outliers. They tested three indication variables via classical examples in both one and two dimensions. The numerical results demonstrate that the new TCIs in general give better results than the original ones and are free of problem-dependent parameters. However, in the two-