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Discrete Morse Flow for Yamabe Type Heat Flows

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Abstract. In this paper, we study the discrete Morse flow for either Yamabe type heat flow or nonlinear heat flow on a bounded regular domain in the whole space. We show that under suitable assumptions on the initial data g one has a weak approximate discrete Morse flow for the Yamabe type heat flow on any time interval. This phenomenon is very different from the smooth Yamabe flow, where the finite time blow up may exist.

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Key Words: Discrete Morse flow; Yamabe type flow; critical exponent; nonlinear heat flow.

1 Introduction

The aim of this note is to develop the discrete Morse flows both for Yamabe type heat flow and for the nonlinear heat flow for any finite time (see [1–3]). The phenomenon for weak solutions is very different from that of the smooth Yamabe flow, where the finite time blowup may exist (see [4] and [2]). We use the idea from [5], where the 2-dimensional Yamabe flow has been studied, to develop the discrete flow. We now recall definition of the weak solution to the Yamabe type heat flow. Let $\Omega\subset R^n$ be a regular bounded domain with smooth boundary. For any $T>0$, we let $Q=Q_T=\Omega\times[0,T]$. Assume that the initial data $g\in C^{2,1}(\overline{Q_T})$ is regular and $g_\nu=0$ on the boundary $\partial\Omega\times\{t\}$, where ν is the outward unit normal to $\partial\Omega$. Assume that $\psi\in C(\overline{\Omega})$ is a non-negative regular function. Hereafter, for a smooth function $u:\overline{\Omega}\to R$, we use the following notations,

$$\nabla u=(\partial u/\partial x^i),\qquad Lu=\Delta u-\psi(x)u.$$

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Assume $n\geq 3$. Recall that for $p=\frac{n+2}{n-2}$, the Yamabe type flow equation is

$$u^{p-1}\partial_t u = Lu, \quad \text{on Q}, \tag{1.1}$$

with the initial data $u=g$ and with the Neumann boundary condition $u_\nu=0$. Let $s_*=\frac{p+1}{2}=\frac{n}{n-2}$, which is the half of the Sobolev critical exponent [6]. We say that the non-negative $u\in C([0,T],H^1(\Omega))$ is a *weak solution* to the Yamabe type flow (1.1) if for any $\eta\in C_0^\infty(Q)$, we have

$$\int_Q \frac{1}{s_*}u^{s_*-1}\partial_t u^{s_*}\eta \mathrm{d}x\mathrm{d}t+\int_Q(\nabla u,\nabla\eta)\mathrm{d}x\mathrm{d}t=\int_Q\psi(x)u\eta \mathrm{d}x\mathrm{d}t$$

and $u(0,\cdot)-g\in H_0^1(\Omega)$ in weak sense that $\lim_{t\to 0}\|u(t,\cdot)-g(\cdot)\|_{L^2}\to 0$. Note that for any $p>1$ being a fixed exponent, one may propose the corresponding nonlinear heat flow and study the weak solution to it.

We have the following conclusion.

Theorem 1.1. *Assume that $\psi=0$ on Ω. For any $T>0$ and any initial-boundary data $g\in C^1(\Omega)$ with $g\geq 0$ and $g_\nu=0$ on the boundary, there exists a discrete Morse flow $\{\hat{u}_N(t)\}$ to the Yamabe type flow* (1.1) *with the initial data $\hat{u}_N(0)=g\geq 0$ and the lateral boundary condition $(\hat{u}_N)_\nu=0$. The limit of the discrete Morse flow is a weak solution to* (1.1) *on* $[0,T]\times\Omega$.

For the precise meaning of the discrete Morse flow, which is the triple $(u_N,\hat{u}_N,\partial_t\hat{u}_N^s)$, one may see the definition under Eq. (2.5) in Section 2. We may use the standard notation $\|u\|_p$ for the norm of the Lebesgue space $L^p(\Omega)$. Other notations are from the famous books [6] and [7].

The use of the discrete Morse flow method (also called Rothe's method) to the study of parabolic problems has a long history and this field is still very active. The discrete Morse flow method was introduced by E.Rothe in the paper [8]. This method for initial boundary value problems, consists of a time variable discretization by finite differences and leads to a sequence of boundary value problems for elliptic equations [9–11]. The method is also known as the horizontal line method for numerical purposes [12]. One may see the book [13] for a friendly introduction of Rothe's method. Only in recent decades, we can see some applications of discrete Morse flows to other geometric flows such as harmonic map heat flows. In [14], the authors applied the discrete Morse flow method to the problem of the heat flow for surfaces of prescribed mean curvature. In the recent work [15], the authors applied the discrete Morse flow method to the parabolic p-Laplacian systems. In the interesting work [16], the discrete Morse flow method had been used to construct infinitely many weak solutions to harmonic map heat flows to spheres. One may also see the references [17–19] for discrete Morse flows for harmonic map heat flows. As one can expect, it is possible to use this method to study weak solutions to Yang-Mills heat flow.

The plan of this paper is below. The main result, Theorem 1.1, will be proved in Section 3. In Section 2, we present the proof of the existence of discrete Morse flow for nonlinear heat flow and the conclusion is stated in Theorem 2.1.