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High Order Conservative Finite Difference/Fourier Spectral Methods for Inviscid Surface Quasi-Geostrophic Flows

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Abstract. In this paper, we develop an effective conservative high order finite difference scheme with a Fourier spectral method for solving the inviscid surface quasi-geostrophic equations, which include a spectral fractional Laplacian determining the vorticity for the transport velocity of the potential temperature. The fractional Laplacian is approximated by a Fourier-Galerkin spectral method, while the time evolution of the potential temperature is discretized by a high order conservative finite difference scheme. Weighted essentially non-oscillatory (WENO) reconstructions are also considered for comparison. Due to a low regularity of problems involving such a fractional Laplacian, especially in the critical or supercritical regime, directly applying the Fourier spectral method leads to a very oscillatory transport velocity associated with the gradient of the vorticity, e.g. around smooth extrema. Instead of using an artificial filter, we propose to reconstruct the velocity from the vorticity with central difference discretizations. Numerical results are performed to demonstrate the good performance of our proposed approach.

AMS subject classifications: 65M06, 65M70, 35R11, 76U60, 76B07, 76B47

Key words: Finite difference, WENO, Fourier spectral, fractional Laplacian, surface quasi-geostrophic flow.

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1 Introduction

In this work, we are interested in the surface quasi-geostrophic (SQG) equations for strong rotating fluids [31, 34, 43]

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$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=0, \tag{1.1a}$$

$$\theta=\left.\frac{\partial\psi}{\partial z}\right|_{z=0}, \quad \lim_{z\to+\infty}\frac{\partial\psi}{\partial z}=0, \tag{1.1b}$$

$$\boldsymbol{u}=\nabla^{\perp}\psi=\left(-\left.\frac{\partial\psi}{\partial y}\right|_{z=0}, \quad \left.\frac{\partial\psi}{\partial x}\right|_{z=0}\right), \tag{1.1c}$$

$$\frac{\partial\theta}{\partial t}+\nabla\cdot(\theta\boldsymbol{u})+\kappa(-\Delta)^s\theta=0. \tag{1.1d}$$

Here $\psi(x,y,z)$ is the potential vorticity, which is uniform due to a zero right side of (1.1a). $\theta(x,y)=\frac{\partial\psi}{\partial z}|_{z=0}$ is the potential temperature (or the buoyancy) restricted on the surface $\mathcal{S}:=\{(x,y,z)\in\mathbb{R}^3:z=0\}$. $\boldsymbol{u}(x,y)$ is the transport velocity determined by the gradient of the vorticity on the surface $\mathcal{S}$, which is the geostrophic balance of rotation and pressure gradients. $\kappa(-\Delta)^s\theta$ accounts for the Ekman pumping effect with a general fractional Laplacian power s and, κ is the Ekman pumping coefficient.

The SQG equations (1.1) can be derived from a three-dimensional (3D) quasi-geostrophic model which describes large-scale mid-latitude atmospheric or oceanographic motions [16, 36, 43]. It is a reduced quasi-geostrophic system for the motion of a stratified fluid in rotation, under a small Rossby number (rotation of the Earth is much larger than the rotation of the motion) and a small Froude number (high stratification) [17, 20, 21, 31, 34, 43, 61]. Inviscid SQG equations with $\kappa=0$ in (1.1d) can be used to simulate atmospheric phenomena, such as frontogeneses, which are strong fronts formed between hot and cold air [20, 43].

We can further reduce the SQG system (1.1) into a two-dimensional (2D) problem by assuming the solution is periodic along the horizontal direction of (x,y). By taking the Fourier transform with respect to (x,y) for (1.1a), first, we obtain

$$\frac{d^2\widehat{\psi}}{dz^2}(\boldsymbol{k},z)=K^2\widehat{\psi}(\boldsymbol{k},z), \tag{1.2}$$

where $\widehat{\psi}(\boldsymbol{k},z)$ is the Fourier transform of ψ with respect to (x,y), $\boldsymbol{k}$ is the horizontal wavenumber and $K=|\boldsymbol{k}|$. Again, by applying the Fourier transform with respect to (x,y) for the two conditions in (1.1b), solving the ordinary differential equation (1.2) with respect to z we get

$$\widehat{\psi}(\boldsymbol{k},z)=-\frac{\widehat{\theta}(\boldsymbol{k})}{K}\exp(-Kz), \tag{1.3}$$

where $\widehat{\theta}(\boldsymbol{k})$ is the Fourier transform of the potential temperature θ. Let $z=0$, by taking the inverse Fourier transform for (1.3), we obtain the following fractional Laplacian equation

$$(-\Delta)^{1/2}\psi(\boldsymbol{x},t)=-\theta(\boldsymbol{x},t). \tag{1.4}$$