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CONSTRUCTION OF BÉZIER SURFACES FROM PRESCRIBED BOUNDARY*

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Abstract

In this paper, we present a method for generating Bézier surfaces from the boundary information based on a general second order functional and a third order functional associated with the triharmonic equation. By solving simple linear equations, the internal control points of the resulting Bézier surface can be obtained as linear combinations of the given boundary control points. This is a generalization of previous works on Plateau-Bézier problem, harmonic, biharmonic and quasi-harmonic Bézier surfaces. Some representative examples show the effectiveness of the presented method.

Mathematics subject classification: 65D17, 65D18.
Key words: Bézier surface, Boundary control points, Quadratic functional, Triharmonic equation.

1. Introduction

As well known, the Bézier model is widely used in CAGD field and geometric modeling because of its excellent properties and algorithms. Using control nets to construct surfaces can better realize interactive design and give the resulting surfaces desired characteristics [1]. Furthermore, it has been a hot topic in CAGD to construct smooth surfaces from different boundary conditions according to different design requirements.

There are various works to generate Bézier surfaces with a wide variety of desired properties. For example, choosing a Bézier surface that verifies a given boundary-value problem such as the standard harmonic and biharmonic PDE enables us to generate a surface which can be controlled solely through the boundary control points [2–6]. The solution of the Plateau-Bézier problem in the case of tensor product Bézier surface is obtained in [7] with the Dirichlet energy instead of the area functional, while harmonic and biharmonic Bézier surface are proposed as an approximation solution of the Plateau-Bézier problem [2]. The minimal quasi-Bézier surfaces in non-polynomial space are also investigated by the Dirichlet method and harmonic method in [8]. Considering the importance of isothermal parameterization in the construction of minimal Bézier surfaces, in [9, 10], we present two new energy functionals called weak-area functional and quasi-area functional respectively by introducing functionals which measure isothermality. A new energy functional called quasi-harmonic energy functional is proposed in [11] as the objective function to obtain the quasi-harmonic Bézier surface from given boundaries. A bending energy for finding Bézier surfaces of minimal bending energy for both triangular and rectangular cases are proposed in [12]. In [13, 14], all the power basis coefficients are obtained

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from boundary control points by using a series of recursive methods of linear equations and then all the control points of the resulting Bézier surface are obtained by the transformation formula between the Bernstein basis and power basis. In [15], Monterde and Ugail present a method to generate Bézier surfaces based on a general 4th-order PDE. A method to construct triharmonic Bézier surfaces with triharmonic equation $\Delta^3\mathbf{X} = 0$ from different boundary conditions is proposed in [1]. In [16], C^1-methods are presented to generate Bézier surfaces from the boundary based on the tetraharmonic equation. Polynomial approximations to minimal Bézier surfaces are discussed in [6], while parametric polynomial minimal surfaces and their properties are presented in [17, 18].

The broad aim of this work is to develop boundary based intuitive surface design techniques for polynomial surfaces particularly for Bézier surfaces which are one of the basic types of surfaces widely used in CAGD. When one is concerned with generating smooth surfaces conforming to a given boundary configuration, it is common to pose the problem within a variational setting [19]. Various functionals can be used for this purpose. This enables us to generate Bézier surfaces with a wide variety of desired properties. We study to construct tensor-product Bézier surface $\mathbf{X}(u, v)$ with a general second order functional and a third order functional related to the triharmonic equation as the objective function respectively from given boundaries. The internal control points of the resulting Bézier surface can be obtained easily by solving simple linear equations.

The paper is organized as follows. In Section 2, we discuss the general quadratic functional and its properties in relation to the corresponding Euler-Lagrange equation. In Section 3 and Section 4, we study the Bézier solutions to the two quadratic functionals respectively from given boundaries and present various examples to show the efficiency of the method. Finally we present a discussion followed by a conclusion in Section 5.

2. Preliminary

In this section, we shall study some quadratic functionals defined on the space of smooth patches $\mathbf{X}$: $\Omega \to \mathbb{R}^3$, where $\Omega = [0, 1] \times [0, 1]$.

Given a Lagrangian

$$T(\mathbf{X}) = T(\mathbf{X}, \mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}, \mathbf{X}_{uv}, \mathbf{X}_{vv}, \mathbf{X}_{uuu}, \mathbf{X}_{uuv}, \mathbf{X}_{uvv}, \mathbf{X}_{vvv}),$$

we take the functional I to be such that,

$$I(\mathbf{X}) = \int_\Omega T(\mathbf{X}) du dv.$$

Minimising the functional I is equivalent to requiring that the first variation of I is zero, which then gives rise to the corresponding Euler-Lagrange equations [15].

Many functionals have been employed in CAGD by several authors, for instance, in [20] it shows that the functional

$$L_1(\mathbf{X}) = \frac{1}{2}\int_\Omega ||\mathbf{X}_{uv}||^2 du dv$$

is related to the Coons patches. The quasi-harmonic functional

$$L_2(\mathbf{X}) = \int_\Omega (||\mathbf{X}_{uu}||^2 + 2 < \mathbf{X}_{uu}, \mathbf{X}_{vv} > + ||\mathbf{X}_{vv}||^2) du dv$$