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Vanishing Viscosity Limit to Planar Rarefaction Wave with Vacuum for 3D Compressible Navier-Stokes Equations

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Abstract. The vanishing viscosity limit of the three-dimensional (3D) compressible and isentropic Navier-Stokes equations is proved in the case that the corresponding 3D inviscid Euler equations admit a planar rarefaction wave solution connected with vacuum states. Moreover, a uniform convergence rate with respect to the viscosity coefficients is obtained. Compared with previous results on the zero dissipation limit to planar rarefaction wave away from vacuum states [27, 28], the new ingredients and main difficulties come from the degeneracy of vacuum states in the planar rarefaction wave in the multi-dimensional setting. Suitable cut-off techniques and some delicate estimates are needed near the vacuum states. The inviscid decay rate around the planar rarefaction wave with vacuum is determined by the cut-off parameter and the nonlinear advection flux terms of 3D compressible Navier-Stokes equations.

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1 Introduction and main results

In the paper, we investigate the vanishing viscosity limit of the three-dimensional (3D) compressible and isentropic Navier-Stokes equations

$$\begin{cases} \rho_t+\operatorname{div}(\rho \boldsymbol{u})=0, \\ (\rho \boldsymbol{u})_t+\operatorname{div}(\rho \boldsymbol{u}\otimes \boldsymbol{u})+\nabla p(\rho)=\mu_1\Delta \boldsymbol{u}+(\mu_1+\lambda_1)\nabla \operatorname{div}\boldsymbol{u}, \end{cases} \tag{1.1}$$

where $\rho=\rho(t,x)\geq 0$, $\boldsymbol{u}=\boldsymbol{u}(t,x)=(u_1,u_2,u_3)(t,x)$ and $p=p(\rho(t,x))$ represent the fluid density, velocity and pressure, respectively, with the spatial variable $x=(x_1,x_2,x_3)\in\Omega\subset\mathbb{R}^3$ and the time variable $t\geq 0$. The pressure $p=p(\rho)$ is given by the well-known γ-law

$$p(\rho)=A\rho^{\gamma} \tag{1.2}$$

with $\gamma>1$ the adiabatic constant and $A>0$ the fluid constant. The two constants μ_1 and λ_1 denote the shear and the bulk viscosity coefficients satisfying the physical restrictions

$$\mu_1>0,\quad 2\mu_1+3\lambda_1\geq 0,$$

and we take

$$\mu_1=\mu'\varepsilon,\quad \lambda_1=\lambda'\varepsilon,$$

where $\varepsilon>0$ is the vanishing parameter, and μ',λ' are the prescribed uniform-in-ε constants.

We consider the viscous system (1.1) in the spatial domain $\Omega=\mathbb{R}\times\mathbb{T}^2$ with $\mathbb{T}^2=(\mathbb{R}/\mathbb{Z})^2$ denoting the two-dimensional unit flat torus, subject to the following initial conditions:

$$(\rho,\boldsymbol{u})(0,x)=(\rho_0,\boldsymbol{u}_0)(x)=(\rho_0,u_{10},u_{20},u_{30})(x) \tag{1.3}$$

with the far fields condition of solutions imposed in the x_1-direction

$$(\rho,u_1,u_2,u_3)(t,x)\ \rightarrow\ (\rho_\pm,u_{1\pm},0,0)\quad \text{as}\quad x_1\rightarrow\pm\infty, \tag{1.4}$$

where $\rho_\pm\geq 0$, $u_{1\pm}$ are the prescribed constants. In the present paper, we are concerned with the case $\rho_-=0$, $\rho_+>0$ such that the end state (ρ_+,u_{1+}) is connected with the vacuum state $\rho_-=0$ through the 2-rarefaction wave.

Formally speaking, as $\varepsilon\rightarrow 0$, the solutions to the 3D compressible Navier-Stokes equations (1.1)-(1.3) with the far fields condition (1.4) converge to the solutions to the 3D compressible Euler equations

$$\begin{cases} \rho_t+\operatorname{div}(\rho \boldsymbol{u})=0,\quad x\in\Omega,\quad t\geq 0, \\ (\rho \boldsymbol{u})_t+\operatorname{div}(\rho \boldsymbol{u}\otimes \boldsymbol{u})+\nabla p(\rho)=0 \end{cases} \tag{1.5}$$