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A FRACTIONAL-ORDER ALTERNATIVE FOR PHASE-LAGGING EQUATION

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Abstract. Phase-lagging equation (PLE) is an equation describing micro/nano scale heat conduction, where the lagging response must be included, particularly under low temperature or high heat-flux conditions. However, finding the analytical or numerical solutions of the PLE is tedious in general. This article aims at seeking a fractional-order heat equation that is a good alternative for the PLE. To this end, we consider the PLE with simple initial and boundary conditions and obtain a fractional-order heat equation and an associated numerical method for approximating the solution of the PLE. In order to better approximate the PLE, the Levenberg-Marquardt iterative method is employed to estimate the optimal parameters in the fractional-order heat equation. This fractional-order alternative is then tested and compared with the PLE. Results show that the fractional method is promising.

Key words. Phase-lagging equation, fractional-order heat equation, numerical scheme, parameter estimation.

1. Introduction

Phase-lagging equation (PLE) is an equation describing micro/nano scale heat conduction, where the lagging response must be included, particularly under low temperature or high heat-flux conditions [1, 2]. The PLE can be expressed as follows [3, 4]:

$$\frac{\partial \theta(\vec{r}, \tau + \tau_0)}{\partial \tau} = D\nabla^2\theta(\vec{r}, \tau), \tag{1}$$

where θ is the temperature, $\vec{r}$ is the position vector, t is the time, D is the thermal diffusivity, τ_0 represents the time lag required to establish steady thermal conduction in a volume element once a temperature gradient has been imposed across it. Based on the Taylor series expression, the zeroth-order approximation of (1) or $\tau_0 = 0$ leads to the common diffusion equation as

$$\frac{\partial \theta(\vec{r}, \tau)}{\partial \tau} = D\nabla^2\theta(\vec{r}, \tau). \tag{2}$$

On the other hand, the first-order approximation of (1) yields a damped wave equation as

$$\frac{\partial \theta(\vec{r}, \tau)}{\partial \tau} + \tau_0 \frac{\partial^2 \theta(\vec{r}, \tau)}{\partial \tau^2} = D\nabla^2\theta(\vec{r}, \tau). \tag{3}$$

By introducing some non-dimensional quantities $u = \theta/\theta_0$, $\vec{x} = \vec{r}/l$, $t = \tau D/l^2$, where $\theta_0 > 0$ is taken as a constant and l is the length of the space domain, the damped wave equation (3) can be transformed in dimensionless form as

$$\frac{\partial u(\vec{x}, t)}{\partial t} + \kappa_0 \frac{\partial^2 u(\vec{x}, t)}{\partial t^2} = \nabla^2 u(\vec{x}, t), \tag{4}$$

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where the dimensionless lag time is given by $\kappa_0 = \tau_0 D/l^2$. Dai and his co-workers [5, 6] compared the difference of the solutions between the PLE and the damped wave equation and showed the damped wave equation is a good approximation for the PLE when the dimensionless time lag κ_0 is small.

It should be pointed out that finding the analytical or numerical solutions of the PLE is tedious in general. Notably, fractional calculus is an emerging field in mathematics with deep applications in all related fields of science and engineering, such as in physics, biology, material science, etc; for example, see [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Recently, some fractional models have been successfully applied to simulate the heat and thermal transfer in non-uniform porous medium, viscoelastic materials, dynamic electro-magnetic fields [18, 19, 20, 21, 22, 23, 24, 25]. This article aims at seeking a fractional-order heat equation that can be a good alternative for the PLE, where the dimensionless time lag κ_0 takes any suitable value, in order to avoid the violation of the second law of thermodynamics [18, 19, 26, 27, 28, 29].

As we know, the fractional calculus possesses the convolution structure, which is similar to the hereditary property of the analytic solution of the PLE (see [5, 6]). This gives us a hint to develop an innovative and accurate fractional-order heat equation to replace the PLE. As such, one could use the solution of the fractional-order heat equation to approximate the solution of the PLE and hence simplify the computation. For this purpose, we propose the fractional-order heat equation

$$\kappa_1 \cdot \frac{\partial u(\vec{x},t)}{\partial t} + \kappa_2 \cdot {}_0^C D_t^{\alpha} u(\vec{x},t) = \nabla^2 u(\vec{x},t), \tag{5}$$

where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1, 2)$, and κ_1 and κ_2 are two constants to be determined, which is a good alternative of (1).

The rest of this paper is organized as follows. In Sect. 2, we consider the fractional-order heat equation and estimate its energy. In Sect. 3, we construct a compact difference scheme for solving the fractional-order heat equation. In Sect. 4, we analyze the unconditional stability and convergence of the scheme rigorously by the discrete energy method. In Sect. 5, we employ the Levenberg–Marquardt iterative method to estimate the parameters of the fractional-order heat equation. In Sect. 6, we compare the solutions between the fractional-order heat equation and the PLE by investigating a testing problem. We summarize the major results of this work in Sect. 7.

2. Fractional-order heat equation

Consider a dimensionless fractional-order heat equation with initial and boundary conditions as follows:

$$\kappa_1 \cdot u_t(x,t) + \kappa_2 \cdot {}_0^C D_t^{1+\beta} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \ (x,t) \in (0,1) \times (0,\infty), \tag{6}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \tag{7}$$

$$u(x,0) = \psi(x), \quad \partial u(x,0)/\partial t = \phi(x), \quad x \in (0,1), \tag{8}$$

where $0 < \beta < 1 (\alpha = 1 + \beta)$, and the Caputo fractional derivative is defined by [8]

$${}_0^C D_t^{1+\beta} u(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u_{ss}(x,s)}{(t-s)^{\beta}} ds, \quad t > 0. \tag{9}$$

Here, we analyze the energy estimation of the model (6)–(8). To this end, we present two useful lemmas with respect to the Caputo factional derivative operator, which will be used for estimating the energy of the governing model (6)–(8).