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Complicated Asymptotic Behavior of Solutions for the Cauchy Problem of Doubly Nonlinear Diffusion Equation

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Abstract. In this paper, we analyze the large time behavior of nonnegative solutions to the doubly nonlinear diffusion equation

$$u_t - \operatorname{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right) = 0$$

in $\mathbb{R}^N$ with $p>1$, $m>0$ and $m(p-1)-1>0$. By using the finite propagation property and the L^1-L^∞ smoothing effect, we find that the complicated asymptotic behavior of the rescaled solutions $t^{\mu/2}u(t^\beta\cdot,t)$ for $0<\mu<2N/(N[m(p-1)-1]+p)$ and $\beta>(2-\mu[m(p-1)-1])/(2p)$ can take place.

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1 Introduction

In this paper, we will study the complicated asymptotic behavior of solutions for the Cauchy problem of the doubly nonlinear diffusion equation

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$$\frac{\partial u}{\partial t}-\mathrm{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right)=0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.1}$$

$$u=u_0 \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $p>1$, $m>0$, $m(p-1)-1>0$ and the nonnegative initial value

$$u_0\in C_0\left(\mathbb{R}^N\right):=\left\{\varphi\in C(\mathbb{R}^N);\lim_{|x|\to\infty}\varphi(x)=0\right\}.$$

In the last few years many authors studied the asymptotic behavior of evolution equations both because many physical phenomena are described by such behavior and because of their mathematical interest (see [4–11, 13, 14, 20, 21, 25, 28]). Among these papers, we mention here an important work by Kamin and Vázquez [13], who reveal that if the initial value

$$u_0\in L^1_+\left(\mathbb{R}^N\right):=\left\{\varphi\in L^1\left(\mathbb{R}^N\right);\varphi(x)\geq 0\right\},$$

then the solutions of Cauchy problem of the porous medium equation (the case $p=2$ of (1.1)) in several dimensions uniformly converge to the Barenblatt solution of the porous medium equation with the same mass as u_0. Later, Zhao and Yuan [28] proved that for the doubly nonlinear diffusion equation (1.1), if the initial value

$$u_0\in L^1_+\left(\mathbb{R}^N\right):=\left\{\varphi\in L^1\left(\mathbb{R}^N\right);\varphi(x)\geq 0\right\},$$

then the solutions of Cauchy problem of (1.1) in several dimensions uniformly converge to the Barenblatt solution with the same mass as u_0.

Unlike the above results that the solutions asymptotic convergence towards a function, Vázquez and Zuazua [20] in 2002 revealed that there exists an initial value $u_0\in L^\infty_+(\mathbb{R}^N)$ such that the set of accumulation points of the rescaled solutions $u(t^{1/2}\cdot,t)$ for the porous medium equation in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ as $t\to\infty$ coincides with the set of $\{S(1)(\varphi)\}$, where φ ranges over the set of the accumulation points as $\lambda\to\infty$ of the family $\{u_0(\lambda x);\ \lambda>0\}$ in the weak-star topology $L^\infty(\mathbb{R}^N)$. That is, the complicated asymptotic behaviors can take place in these equations. Later Cazenave, Dickstein, Weissler and Fang investigated the complicated asymptotic behavior of the rescaled solutions $t^{\mu/2}u(t^\beta x,t)$ $(\mu,\beta>0)$ for heat equation in [5–7, 9, 10, 14] and for the Navier-Stokes equations in [8] and for Schrödinger equation in [11]. And in 2007, Carrillo and Vázquez [4] show that if the initial value $u_0\in L^1_+(\mathbb{R}^N)\bigcap L^\infty(\mathbb{R}^N)$ with finite second moment, then the complicated asymptotic behavior of solutions for some filtration equations

$$\frac{\partial u}{\partial t}-\Delta\Phi(u)=0$$