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Within-Cluster Variability Exponent for Identifying Coherent Structures in Dynamical Systems

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Abstract. We propose a clustering-based approach for identifying coherent flow structures in continuous dynamical systems. We first treat a particle trajectory over a finite time interval as a high-dimensional data point and then cluster these data from different initial locations into groups. The method then uses the normalized standard deviation or mean absolute deviation to quantify the deformation. Unlike the usual finite-time Lyapunov exponent (FTLE), the proposed algorithm considers the complete traveling history of the particles. We also suggest two extensions of the method. To improve the computational efficiency, we develop an adaptive approach that constructs different subsamples of the whole particle trajectory based on a finite time interval. To start the computation in parallel to the flow trajectory data collection, we also develop an on-the-fly approach to improve the solution as we continue to provide more measurements for the algorithm. The method can efficiently compute the WCVE over a different time interval by modifying the available data points.

AMS subject classifications: 37M05, 37M10, 37M25, 65L05

Key words: Dynamical system, visualization, finite time Lyapunov exponent, numerical methods for differential equations, k-means clustering.

1 Introduction

Finite-time Lyapunov exponent (FTLE) [7–9,11,18] is a widely used Lagrangian quantity to hint the location of any Lagrangian coherent structure (LCS) in a given velocity field $\mathbf{u}(\mathbf{x}(t),t)$. It measures the rate of change in the distance between neighboring particles across a finite interval of time with an infinitesimal perturbation in the initial position. To obtain the FTLE field, one needs to first compute the flow map which links the initial

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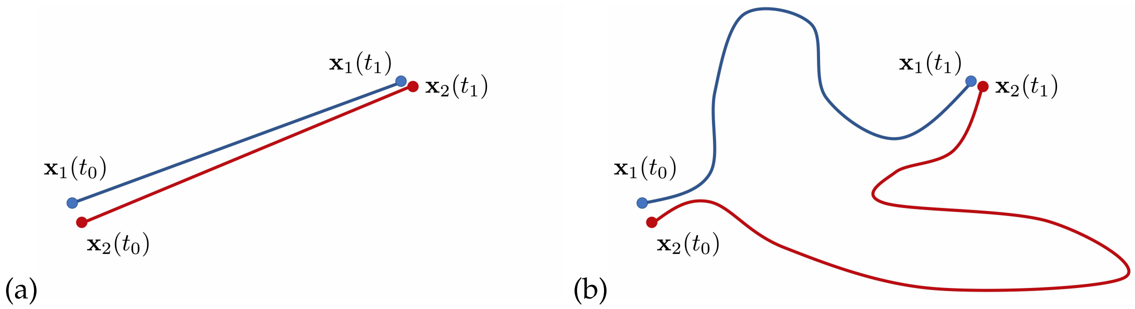

Figure 1: FTLE requires only the flow map from the initial location at the initial time to the final arrival location at the terminal time but ignores the particle's intermediate locations.

location of a particle with the arrival position based on the characteristic line. Mathematically these particles in the extended phase space satisfy the ordinary differential equation (ODE) given by

$$\dot{\mathbf{x}}(t)=\mathbf{u}(\mathbf{x}(t),t) \tag{1.1}$$

with the initial condition $\mathbf{x}(t_0)=\mathbf{x}_0$ and a Lipschitz velocity field $\mathbf{u}:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$. The flow map $\Phi_{t_0}^{t_0+T}:\mathbb{R}^d\to\mathbb{R}^d$ is defined as the mapping which takes the point $\mathbf{x}_0$ to the particle location at the final time $t=t_0+T$, i.e $\Phi_{t_0}^{t_0+T}(\mathbf{x}_0)=\mathbf{x}(t_0+T)$ with $\mathbf{x}(t)$ satisfying (1.1). The FTLE is then defined using the largest eigenvalue of the deformation matrix based on the Jacobian of this resulting flow map. In a series of recent studies [12–15, 23–25, 28], we have developed various Eulerian approaches to numerically compute the FTLE on a fixed Cartesian mesh. The idea is to combine the approach with the level set method [16,17] which allows the flow map to satisfy a Liouville equation. Such hyperbolic partial differential equations (PDEs) can then be solved by any well-established robust and high order accurate numerical methods. We have also developed several other quantities to extract coherent structures in a given velocity field. But we will not compare them with the proposed approach in this work but concentrate only on the FTLE as a demonstration. We refer interested readers to [26,27] and references therein for more discussions.

From the computation of the FTLE, we notice that FTLE ignores the particle's intermediate locations, which might lead to a misinterpretation of coherent structure in a dynamic system. Because the FTLE requires only the flow map from the initial location at the initial time to the final arrival location at the terminal time, the quantity cares only about the particle's initial and final positions. If an initial patch of particles stays close for a significant period but diverts quickly right before the terminal time, FTLE would still give a large quantity. Suppose particles in an initial patch travel very differently with unrelated traveling history but reach a similar region at the terminal time. As shown in Fig. 1, FTLE ignores any significant dispersion at the intermediate time and returns a small quantity. When neighboring particles arrive at the exact locations, all intermediate locations are irrelevant to the FTLE completely. Therefore, the FTLE for both cases