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A NEW FINITE ELEMENT SPACE FOR EXPANDED MIXED FINITE ELEMENT METHOD *

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Abstract

In this article, we propose a new finite element space $\mathbf{\Lambda}_h$ for the expanded mixed finite element method (EMFEM) for second-order elliptic problems to guarantee its computing capability and reduce the computation cost. The new finite element space $\mathbf{\Lambda}_h$ is designed in such a way that the strong requirement $\mathbf{V}_h \subset \mathbf{\Lambda}_h$ in [9] is weakened to $\{\mathbf{v}_h \in \mathbf{V}_h; \mathrm{div}\mathbf{v}_h = 0\} \subset \mathbf{\Lambda}_h$ so that it needs fewer degrees of freedom than its classical counterpart. Furthermore, the new $\mathbf{\Lambda}_h$ coupled with the Raviart-Thomas space satisfies the inf-sup condition, which is crucial to the computation of mixed methods for its close relation to the behavior of the smallest nonzero eigenvalue of the stiff matrix, and thus the existence, uniqueness and optimal approximate capability of the EMFEM solution are proved for rectangular partitions in $\mathbb{R}^d, d = 2, 3$ and for triangular partitions in $\mathbb{R}^2$. Also, the solvability of the EMFEM for triangular partition in $\mathbb{R}^3$ can be directly proved without the inf-sup condition. Numerical experiments are conducted to confirm these theoretical findings.

Mathematics subject classification: 65N30.
Key words: New finite element space, Expanded mixed finite element, Minimum degrees of freedom, The inf-sup condition, Solvability, Optimal convergence.

1. Introduction

The expanded mixed finite element method (EMFEM) [9], first proposed for linear elliptic problems of second-order to generalize the classical mixed methods in the sense that the gradient as a newly introduced variable is explicitly approximated besides the unknown and flux, has achieved a significant success in applications to those diffusion processes within complex geometry and low permeability zones. Now the EMFEM has been extended successively to the quasi-linear elliptic problems [10, 21], the fourth order elliptic equations [10], parabolic problems [8, 16, 19, 24], hyperbolic problems [27], displacement in porous media [18, 28] and other physical models [6, 17, 20]. Recently, the EMFEM was found its application to the fractional-order diffusion equations [7, 26].

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Checking carefully the direct proof for the EMFEM's solvability (see pp. 487 in [9]), it is not difficult to find that the finite element space $\mathbf{\Lambda}_h$ for the gradient and the space $\mathbf{V}_h$ for the flux must satisfy the strong requirement $\mathbf{V}_h \subset \mathbf{\Lambda}_h$, which makes the selection of the test function $\boldsymbol{\mu}_h = \boldsymbol{\sigma}_h \in \mathbf{V}_h$ possible to ensure the existence and uniqueness of the solution to the EMFEM. As a consequence, the strong requirement $\mathbf{V}_h \subset \mathbf{\Lambda}_h$ excludes such the potential spaces as those $\mathbf{\Lambda}_h$ with lower space indices, especially the often used piecewise-constant spaces. This may confine the versatility of the EMFEM in applications and increase the computing burden. For example, if $\mathbf{V}_h$ and W_h are the the lowest order Raviart-Thomas space for triangular partitions, the best convergence rates for the unknown in $L^2(\Omega)$-norm, the gradient in $(L^2(\Omega))^d$-norm and the flux in $\mathbf{H}(\mathrm{div};\Omega)$-norm are $\mathcal{O}(h)$ whatever the space index of $\mathbf{\Lambda}_h$ takes 1 or 0.

The main goals of this article are to: (1) Redesign the finite element space $\mathbf{\Lambda}_h$ in such a way that they contain as many full polynomials as W_h does in order to preserve the same approximate capability, and contain all the divergence-free vectors of $\mathbf{V}_h$ to ensure the solvability of the EMFEM for linear elliptic problems of second-order, and hence the strong requirement $\mathbf{V}_h \subset \mathbf{\Lambda}_h$ is weakened and thus designed $\mathbf{\Lambda}_h$ possesses minimum degrees of freedom. Specifically, for rectangular partitions, the degrees of freedom of $\mathbf{\Lambda}_h$ on an element E are $2k+1$ and $2(k+1)^2 - 1$ degrees of freedom less than those of $\mathbf{V}_h|_E$ for $d = 2$ and $d = 3$, respectively; for triangular partitions, $\mathbf{\Lambda}_h$ consists of all the piecewise polynomials of degree $\leq k$, which are $k+1$ and $\frac{1}{2}(k+1)(k+2)$ degrees of freedom less than that of $\mathbf{V}_h|_E$ for $d = 2$ and $d = 3$, respectively. And thus, the commonly used piecewise constant spaces are retrieved. (2) Prove that thus redesigned $\mathbf{\Lambda}_h$ combined with the Raviart-Thomas mixed space $\mathbf{V}_h \times W_h$ satisfies the coerciveness condition and the inf-sup condition. This finding is crucial to the computation of mixed methods since the inf-sup condition is closely related to the behavior of the smallest nonzero eigenvalue of the stiff matrix, the loss of which may leads extra artificial (nonphysical) constraints on the boundary conditions or locking phenomenon [3]. (3) Prove the existence, uniqueness and the same approximate capability of the EMFEM solution as the traditional mixed methods [9,10], by an application of the coerciveness and the inf-sup condition for rectangular partitions of $\Omega \subset \mathbb{R}^d, d = 2, 3$ and triangular partitions of $\Omega \subset \mathbb{R}^2$. (4) Present a direct proof as did in [9] for the solvability of the EMFEM on general partitions of $\mathbb{R}^d$. (5) Conduct numerical experiments to confirm the theoretical findings.

The rest of this article is organized as follows. In Section 2, we shall develop the weak form and the EMFEM for linear elliptic problems, prove the coerciveness for one of bilinear form and analyze the key points we will stress in the sequel. Section 3 is devoted to the rectangular partitions for $\Omega \subset \mathbb{R}^d$. we shall characterize the divergence-free vectors of $\mathbf{V}_h$ by decomposition techniques, redesign the space $\mathbf{\Lambda}_h$ with minimum degrees of freedom, then prove the validation of the inf-sup condition for the other bilinear form over the $\mathbf{\Lambda}_h$ and the Raviart-Thomas space. Section 4 is devoted to the triangular partitions. We shall use the inclusion of divergence-free vectors of $\mathbf{V}_h$ to redesign the $\mathbf{\Lambda}_h$ and give a direct proof for the solvability of the EMFEMs for $d = 2$ and $d = 3$. Further, we shall apply the discrete Helmholtz decomposition theory to prove the validation of the inf-sup condition over the newly defined space $\mathbf{\Lambda}_h$ and the $\mathbf{V}_h \times W_h$ for $d = 2$, then derive the solvability again and the same approximate capability of the EFEMs solution as that of the traditional mixed methods have. In Section 5, numerical experiments are conducted to confirm our theoretical findings. The last section is for concluding remarks.

Through out this paper, we write vectors or vector spaces in boldface, use $(\cdot,\cdot)$ to denote the L^2-inner product, and use $\|\cdot\|$ to denote the L^2-norm or the Euclid norm in vector spaces. We also use $\|\cdot\|_H$ to denote the norm in Sobolev space H and $|\cdot|_H$ to denote its semi-norm.