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OPTIMAL CONTROL FOR MULTISCALE ELLIPTIC EQUATIONS WITH ROUGH COEFFICIENTS*

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Abstract

This paper concerns the convex optimal control problem governed by multiscale elliptic equations with arbitrarily rough L^∞ coefficients, which has not only complex coupling between nonseparable scales and nonlinearity, but also important applications in composite materials and geophysics. We use one of the recently developed numerical homogenization techniques, the so-called Rough Polyharmonic Splines (RPS) and its generalization (GRPS) for the efficient resolution of the elliptic operator on the coarse scale. Those methods have optimal convergence rate which do not rely on the regularity of the coefficients nor the concepts of scale-separation or periodicity. As the iterative solution of the nonlinearly coupled OCP-OPT formulation for the optimal control problem requires solving the corresponding (state and co-state) multiscale elliptic equations many times with different right hand sides, numerical homogenization approach only requires one-time pre-computation on the fine scale and the following iterations can be done with computational cost proportional to coarse degrees of freedom. Numerical experiments are presented to validate the theoretical analysis.

Mathematics subject classification: 65M06, 65M12, 65M15, 65M55.
Key words: Optimal control, Rough coefficients, Multiscale elliptic equations, Numerical homogenization, Rough polyharmonic splines, Iterative algorithm.

1. Introduction

In the optimal control problem, the state variable y is a model of a system whose evolution can be influenced through the control variable u. The relations between u and y are given by the state equation. The optimal control variable u is found by minimizing the objective functional.

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In this paper, we consider the following convex optimal control problem (OCP) governed by elliptic partial differential equations

$$\min_{u\in K\subset L^2(\Omega_U)} g(y)+h(u)$$
$$\text{subject to} \quad \begin{cases} -\operatorname{div}(a(x)\nabla y)=f(x)+Bu, & \text{in } \Omega, \\ y=0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $g(\cdot), h(\cdot)$ and $f(\cdot)$ are given functions, and B is a given operator. In this paper, we consider the case when the rough coefficient $a(x)$ exhibits multiscale behaviour. In particular, we consider the optimal control problem where the spatial variation of the diffusion coefficient is on a fine scale compared to the computational domain. Numerically approximating the solutions in the usual way is difficult, because a very fine discretization is necessary to resolve the fine-scale structure.

Optimal control problem plays an increasingly important role in many engineering branches, and efficient numerical methods are essential to its successful applications [1,2]. Over the past 30 years, finite element method (FEM) has become one of the most widely used numerical methods for optimal control problems. For the optimal control of elliptic or parabolic equations: a priori error estimates [1,3,4], a posteriori error estimates [5,6], and some superconvergence results [7,8] for the FEM methods have been developed in the literature. However, those error estimates require the H^2 regularity of the solutions.

The optimal control problems governed by partial differential equations with rough coefficients (such as permeabilities in reservoir modelling) have become a great challenge, owing to the lack of regularity of the coefficients $a(x)$ and the solutions $y(x)$ and $p(x)$ ($p(x)$ is the co-state variable in (4.2)). The solution and co-state variable are in $H^1(\Omega)$ but not in $H^2(\Omega)$. Even if $a(x)$ is a smooth but highly oscillatory function in the form of $a(x, x/\varepsilon)$ with a small parameter $\varepsilon \ll 1$ (such as material properties of composite materials), conventional FEMs based on piecewise polynomial basis become ineffective [9]. To be more precise, the convergence of conventional FEMs relies on the H^2-regularity of the solution, but the prefactor of the error is of the order $O(1/\varepsilon)$. Thus, conventional FEMs require prohibitively small mesh size $h < \varepsilon$ to yield good numerical approximations for the optimal control problem.

Numerical homogenization for problems with multiple scales have attracted increasing attention in recent years. It is motivated by the fact that standard methods, such as finite-element method with piecewise linear elements [10] can perform arbitrarily badly for PDEs with rough coefficients. If the coefficient $a(x)$ has structural properties such as scale separation and periodicity, together with some regularity assumptions (e.g., $a(x) \in W^{1,\infty}$), classical homogenization [11, 12] can be used to construct efficient multiscale computational methods and have been applied to optimal control problems, such as multiscale asymptotic expansions method [13–15], multiscale finite element method (MsFEM) [9,16–19], heterogeneous multiscale method (HMM) [20–23], and Localized Orthogonal Decomposition (LOD) method [24–26].

For multi-scale PDEs with non-separable scales and high-contrast coefficients which appear in many applications such as the optimal design of composite materials and the control of water injection in reservoir simulation, the coefficients do not have structural properties such as periodicity/scale separation. Numerical homogenization with non-separable scales concerns approximation of the solution space for such problems by a (coarse) finite dimensional space, instead of focusing on the classical issue of the homogenization limit. Fundamental questions for numerical homogenization are: How to approximate the high dimensional solution space by a low dimensional approximation space with optimal error control, and furthermore, how to