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A Reconstructed Discontinuous Approximation to Monge-Ampère Equation in Least Square Formulation

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Abstract. We propose a numerical method to solve the Monge-Ampère equation which admits a classical convex solution. The Monge-Ampère equation is reformulated into an equivalent first-order system. We adopt a novel reconstructed discontinuous approximation space which consists of piecewise irrotational polynomials. This space allows us to solve the first-order system in two sequential steps. In the first step, we solve a nonlinear system to obtain the approximation to the gradient. A Newton iteration is adopted to handle the nonlinearity of the system. The approximation to the primitive variable is obtained from the approximate gradient by a trivial least squares finite element method in the second step. Numerical examples in both two and three dimensions are presented to show an optimal convergence rate in accuracy. It is interesting to observe that the approximation solution is piecewise convex. Particularly, with the reconstructed approximation space, the proposed method numerically demonstrates a remarkable robustness. The convergence of the Newton iteration does not rely on the initial values. The dependence of the convergence on the penalty parameter in the discretization is also negligible, in comparison to the classical discontinuous approximation space.

AMS subject classifications: 65N30

Key words: Monge-Ampère equation, least squares method, reconstructed discontinuous approximation.

1 Introduction

The elliptic Monge-Ampère equation is a fully nonlinear second-order partial differential equation, which arises naturally from the geometric surface theory and from the applications such as optimal mass transportation, kinetic theory, geometric optics, image processing and others, and we refer to [13, 14, 25] and the references therein for an

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extensive review of applications. Recently, the numerical scheme for solving the elliptic Monge-Ampère equation has been a subject of particular interests [4]. It is known that if the classical solution exists the solution to the Monge-Ampère equation is strictly convex on the smooth domain with the positive source term. Hence, the Monge-Ampère equation is challenging to solve numerically due to its full nonlinearity and the convex constraint. We refer to the review papers [21, 35] for an overview of the numerical challenges and the history of the work on this problem.

In 1988, Prussner and Oliker introduced a finite difference scheme in [37] for the Monge-Ampère equation. The discretization was based on the geometric interpretation of the equation and they proved that the method converges to the generalized solution in two dimensions. Froese and Oberman proposed a convergent monotone finite difference scheme by constructing a wide stencil. We refer to [3, 16, 24, 36] for more discussion and some improvements on the wide stencil scheme. Another simple finite difference method was proposed in [4] but the proof of convergence remains an open problem. Galerkin-type methods have also been investigated for the Monge-Ampère equation and an immediate challenge is the problem does not naturally fit within the Galerkin framework [16]. Böhmer introduced an L^2 projection method in [6] by applying the C^1 finite element spaces. Brenner et al. [7] proposed a C^0 finite element method. They proposed a discrete linearization which is consistent with continuous linearization. Dean and Glowinski [19, 20] reformulated the Monge-Ampère equation as a minimization problem by applying the augmented Lagrangian method. The minimization problem can then be solved with mixed finite element methods. Feng and Neilan added a small multiple of the biharmonic operator to the Monge-Ampère equation. The resulted fourth-order PDE is solved by mixed finite element methods [22, 23]. Besides, there are some least squares finite element methods proposed for the Monge-Ampère equation and we refer to [9, 10, 40].

In this paper, we propose a new least squares finite element method for solving the Monge-Ampère equation with classical solutions. As a preparation, we reformulate the Monge-Ampère equation into an equivalent first-order system and we solve the first-order problem in two sequential steps. In the first step, we solve a nonlinear first-order system to obtain the approximation to the gradient by a piecewise irrotational polynomial space. This space is obtained by the patch reconstruction process with only one unknown per element [27, 29, 39]. The second step is to solve a linear first-order system to seek a numerical approximation to the primitive variable.

The numerical scheme to the first nonlinear problem is the main component in our method. We apply the standard Newton-type linearization to the nonlinear operator, and from the linearization we formally give the non-divergence problem in each iteration step. In the discrete level, given a numerical approximation, we solve the non-divergence form problem by minimizing a least squares functional on the reconstructed space, and then update the numerical approximation for the next step via the Newton iterative method. In the second step, we introduce another least squares functional to solve the linear problem. This functional is then minimized in the Lagrange finite element space, together with the numerical gradient from the first step, to seek a nu-