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An Intrinsic Formulation of the von Kármán Equations

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Abstract. To begin with, we identify the intrinsic equations of a von Kármán nonlinearly elastic plate, which allow to directly compute the stresses inside the plate without having to first compute the displacement field, by contrast with the classical displacement approach. Then we establish that these intrinsic equations possess weak solutions, which are the bending moments and stress resultants of the middle surface of the plate.

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1 Introduction

When an elastic plate is subjected to specific boundary conditions and applied forces, its reference configuration, i.e., the portion of space it occupies in the absence of forces, becomes a deformed configuration. One of the central themes of plate theory then consists in providing equations that allow to determine the displacement vector at each point of the reference configuration. The unknown is thus a vector field defined over the reference configuration, whose components are those of the unknown displacement field. These equations originally took the

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form of boundary value problems, i.e., partial differential equations on the middle surface of the plate, complemented by suitable boundary conditions. However, it was subsequently recognised that minimising an ad hoc energy functional over an appropriate set of admissible displacements of the middle surface of the plate was the most efficient way to obtain existence theorems.

From the computational viewpoints, however, this classical approach is not fully satisfactory, in that the unknowns of primary interest in practical applications are not so much the components of the displacement field, but instead those of the stress tensor field inside the body, since large stresses, rather than large displacements, are more likely to provoke the collapse of an elastic structure. But computing the stresses from the displacement, by means of the constitutive equation of the elastic material constituting the structure, involves computing derivatives, a procedure well-known to be unstable numerically.

By contrast, in an intrinsic approach, it is the components of the stress tensor, or more generally of any bona fide measure of stress, that are the only unknowns, instead of the components of the displacement vector field. Although he idea of using such intrinsic approaches for the modelling of elastic bodies goes back to Chien [6, 7] and, more recently, to Antman [1], it is only in the last two decades that mathematical foundations of such methods have been investigated, first for the three-dimensional model of elasticity, both linear and nonlinear (see [10]), then for two-dimensional models for plates and shells, but only for linear models (see [11, 12]), or for the nonlinear Kirchhoff-Love model (see [13–15]).

The objective of this paper is to address one of the missing cases, namely the von Kármán equations for a nonlinearly elastic plate (see [20]). The new nonlinear plate model, which is defined in Theorem 4.1 below, is equivalent to von Kármán's model, but has the advantage of being defined solely in terms of the bending moments and stress resultants of the middle surface of the plate, instead of the vertical component of the displacement field and of the Airy function in the original von Kármán equations. This is of importance in applications where lower-dimensional models for thin elastic bodies are most of the time used to predict the stresses that may appear in them (see, e.g., [17–19]).

The paper is organized as follows. In the Section 2, we first introduce the notation and all relevant notions from differential geometry of surfaces that we will need. Then we describe the type of applied forces and boundary conditions that correspond to the von Kármán equations.

In Section 3, we state the classical two-dimensional von Kármán equations for a nonlinearly elastic plate and give a brief, but self-contained, account of the derivation of these equations as the limit as the thickness of the plate approaches zero of the three-dimensional equations of the plate.