Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 5, Number 3, September 2023, 621–636 DOI:10.12150/jnma.2023.621

Zero-Hopf Bifurcation at the Origin and Infinity for a Class of Generalized Lorenz System*

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Abstract In this paper, the zero-Hopf bifurcations are studied for a generalized Lorenz system. Firstly, by using the averaging theory and normal form theory, we provide sufficient conditions for the existence of small amplitude periodic solutions that bifurcate from zero-Hopf equilibria under appropriate parameter perturbations. Secondly, based on the Poincaré compactification, the dynamic behavior of the generalized Lorenz system at infinity is described, and the zero-Hopf bifurcation at infinity is investigated. Additionally, for the above theoretical results, some related illustrations are given by means of the numerical simulation.

Keywords Generalized Lorenz system, zero-Hopf bifurcation, averaging theory, normal form theory, Poincaré compactification

MSC(2010) 34C05, 37C07.

1. Introduction

Due to the great application of chaotic systems in the real world, more and more scholars have focused on the complex dynamic properties of the chaotic models and the generation mechanism of chaos. Many different types of chaotic models have been found or constructed after Lorenz model was presented [16], for instance, Chua's circuit system [4], Chen system [3], Lü system [18], Yang-Chen system [26] and other Lorenz-type systems [9,13]. Then a great number of results on theoretical analysis have been obtained for the practical chaotic models above mentioned in the past several decades.

Here it is worth mentioning the study on Hopf bifurcation for these chaotic models. Its early results put emphasis on the existence and stability of only a single Hopf bifurcation, see e.g., [7, 10]. And then the multiple Hopf bifurcations began to be investigated for some models, for example, the Lü system [19], the Lorenz system [23], and the Maxwell-Bloch system [11]. The readers can also refer to the

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*This work was partially supported by Natural Science Foundation of China (Nos.12161023, 12061016), Nature Science Foundations of Guangxi (No.2020GXNSFAA159138), School graduate program (2021YJSPY01)

recent literatures [5, 17, 21]. However, for many chaotic models as general high-dimensional systems, the upper bounds on the cyclicity in the vicinity of a Hopf singular point is a very challenging problem [24, 28].

As for the zero-Hopf bifurcation, recently it is getting more and more attention in the researches on chaotic models. One important reason is that the zero-Hopf singular point of the high-dimensional system may reflect the emergence of chaotic behavior [1]. The hallmark of zero-Hopf bifurcation is that the linear part of the system has a zero eigenvalue and a pair of pure conjugates complex eigenvalues. The common tool for investigating this problem is the averaging theory, and many chaotic systems were considered, e.g., [14, 20]. Furthermore, the authors of [30] applied the normal form theory to investigate Rössler system, and showed that the method of normal forms is applicable for all types of zero-Hopf bifurcations, while the averaging method is successful only for a certain type of zero-Hopf singular points.

In addition, the study on the zero-Hopf bifurcation in some chaotic models can extend to two aspects: one is its multiplicity and cyclicity, and the other is zero-Hopf bifurcation problem at infinity. For the former, it is less fully studied though multiple limit cycles have been discovered in the application of averaging theory of second order, see e.g. [20]. For the latter, it has also been rarely considered though many dynamic behaviors at infinity have begun to be intensively analyzed [6, 12, 15, 25].

In this paper, we will consider a generalized Lorenz system which the following form

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - y - xz + dyz + ey^2 + fxy, \\ \dot{z} = xy - bz, \end{cases} \tag{1.1}$$

where $a, b, c, d, e, f \in \mathbb{R}$. When $d = e = f = 0$, system (1.1) becomes the Lorenz system. It is different from the generalized Lorenz system early proposed in [2], but system (1.1) also contains all the information of the Lorenz system. Here we will study the zero-Hopf bifurcation at finite equilibria and infinity for this generalized Lorenz system (1.1), and try to determine the zero-Hopf cyclicity only in the sense of first order averaging.

The rest of the article is organized as follows. In Section 2, we study the zero-Hopf bifurcations at the finite equilibria by using the averaging theory and normal form theory successively for system (1.1). In Section 3, the dynamical behaviors at infinity are discussed via the Poincaré compactification of the polynomial vector field in $\mathbb{R}^3$. In particular, the zero-Hopf bifurcation at infinity is investigated by applying the normal form theory. For the theoretical results obtained, the corresponding numerical simulations are given respectively in the above process of analysis.

2. Zero-Hopf bifurcation for the equilibria

In this section, we will study zero-Hopf bifurcation at finite equilibrium points. System (1.1) always has the equilibrium point $O = (0, 0, 0)$ for any parameter