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Monotonic Diamond and DDFV Type Finite-Volume Schemes for 2D Elliptic Problems

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Abstract. The DDFV (Discrete Duality Finite Volume) method is a finite volume scheme mainly dedicated to diffusion problems, with some outstanding properties. This scheme has been found to be one of the most accurate finite volume methods for diffusion problems. In the present paper, we propose a new monotonic extension of DDFV, which can handle discontinuous tensorial diffusion coefficient. Moreover, we compare its performance to a diamond type method with an original interpolation method relying on polynomial reconstructions. Monotonicity is achieved by adapting the method of Gao *et al* [A finite volume element scheme with a monotonicity correction for anisotropic diffusion problems on general quadrilateral meshes] to our schemes. Such a technique does not require the positiveness of the secondary unknowns. We show that the two new methods are second-order accurate and are indeed monotonic on some challenging benchmarks as a Fokker-Planck problem.

AMS subject classifications: 65N08, 65N12
Key words: Finite volume method, anisotropic diffusion, monotonic method, DDFV scheme.

1 Introduction

Consider the model stationary diffusion problem

$$\begin{cases} -\boldsymbol{\nabla}\cdot(\boldsymbol{\kappa}\boldsymbol{\nabla}\bar{u})+\lambda\bar{u}=f & \text{in } \Omega, \\ \bar{u}=g & \text{on } \Gamma_D, \\ \boldsymbol{\kappa}\boldsymbol{\nabla}\bar{u}\cdot\mathbf{n}=g & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

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where Ω is a bounded open domain of $\mathbb{R}^2$ with $\partial\Omega = \Gamma_D \cup \Gamma_N$ ($\Gamma_D \cap \Gamma_N = \emptyset$) and $\mathbf{n} \in \mathbb{R}^2$ the outgoing unit normal vector. The data are such that $f, \lambda \in L^2(\Omega)$, with $\lambda \geq 0$, $g \in H^{1/2}(\Gamma_D)$ and $g \in L^2(\Gamma_N)$. The tensor-valued diffusion coefficient $\boldsymbol{\kappa}$ is supposed to be bounded and to satisfy the uniform ellipticity condition

$$\forall \mathbf{x} \in \Omega, \quad \forall \mathbf{y} \in \mathbb{R}^2, \quad \alpha_{\min} \|\mathbf{y}\|^2 \leq \mathbf{y}^t \boldsymbol{\kappa}(\mathbf{x}) \mathbf{y} \leq \alpha_{\max} \|\mathbf{y}\|^2,$$

where $\alpha_{\min}$, $\alpha_{\max}$ are positive coefficients. Under the above conditions, and if either $\lambda > 0$ or Γ_D is of positive length, it is well known that system (1.1) has a unique solution in $H^1(\Omega)$. Such a solution satisfies a positiveness principle, i.e. if $f \geq 0$ and $g \geq 0$, then $\bar{u} \geq 0$ (see [15] for example).

Standard methods may be applied to the discretization of such diffusion equations with possibly discontinuous κ on arbitrary meshes. This proves to be an efficient strategy, as far as accuracy (or convergence) is concerned. However, it is well known that positiveness of the discrete solution does not hold. This lack of positiveness (also called monotonicity) can lead to serious difficulties, since $\bar{u}$ can account for a temperature or a concentration. A first attempt to solve the issue of monotonicity would be to truncate the discrete solution to zero. This is not satisfactory because conservation is lost in such a process, and conservation is an important property of the scheme. Some algorithms based on the repair technique introduced in [34] are employed to fix the conservation issue [8, 33, 43, 45]. However, these algorithms are only *globally* (and not locally) conservative, and the consistency is unclear. Some monotonic methods have been designed in the finite-element framework (see [9,10,24,25,41] among others), but they rely on restrictive conditions on the mesh, that we cannot afford.

For fifteen years many original finite volume methods have been proposed to address the issue of monotonicity, while preserving conservation. Most of these schemes are nonlinear or have a larger stencil than standard methods. The finite volume framework is well suited to achieve monotonicity because it allows for an easy manipulation of the fluxes. The first works we know of are those of Le Potier [28] and Bertolazzi and Manzini [3]. In such methods, one uses a manipulation of the fluxes that leads to introduce a dependence on the discrete solution in the coefficients of the fluxes, making the scheme nonlinear, although (1.1) is linear. To this end, one usually introduces secondary unknowns (for instance vertex-located or face-located unknowns) in addition to the primary (cell-located) unknowns. Among others, important contributions to this field are [5, 18, 30, 39, 48], which propose efficient numerical schemes preserving the positiveness of the primary unknowns. In [38] the requirement of positive secondary unknowns is relaxed. The works [31, 50] explain how to build monotonic schemes without relying on secondary unknowns. In [29, 32, 37], maximum principle preserving schemes are proposed. Cancès and Guichard obtained moreover an entropy diminishing property in [7], introducing the nonlinearity directly at the continuous level via a change of variables. Some concepts and proofs about the existence of solutions for these types of scheme can be found in [11, 13, 36]. See also [42, 46] for recent advances in this field.