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Global Solutions of Nematic Liquid Crystal Flow in Dimension Two

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Abstract. In this article we are concerned with a simplified Ericksen-Leslie system on $\mathbb{R}^2$, whose bounded domain case was considered in [Lin *et al.*, Arch. Ration. Mech. Anal. 197 (2010), 297–336]. With a study of its vorticity-stream formulation, we establish a global existence result of weak solutions when initial orientation has finite energy and initial vorticity function lies in $L^1(\mathbb{R}^2)$.

AMS subject classifications: 35A01, 35D30

Key words: Ericksen-Leslie system, circulation Reynolds number, vorticity formulation.

1 Introduction

1.1 Background and motivation

Ericksen-Leslie system is a hydrodynamic system modeling the flow of nematic liquid crystals. Proposed in [7, 19] and references therein, it is a continuum theory without molecular details of a liquid crystal material. Recently some research works have been devoted to studying the relationships between the theory of Ericksen-Leslie and two other favorable theories (Doi-Onsager theory and Landau-de Gennes theory) for nematic liquid crystals. In [30] the Doi-Onsager

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theory (see [6,25]) is connected with the Ericksen-Leslie theory by taking the Deborah number to zero. As a hydrodynamic Landau-de Gennes model, the Beris-Edwards system (see [4]) was studied by the authors in [26, 27, 29]. Particularly in [30], a Hilbert expansion was obtained for solutions of the Beris-Edwards system with which a well prepared initial data is supplied. When elastic constants are small, their work rigorously shows that the Ericksen-Leslie system serves as the limit of the Beris-Edwards system before the first singular time. For the static theory of liquid crystals, readers should be referred to [1, 24] for important connections and differences between the Landau-de Gennes theory and the Oseen-Frank theory. As far as the Ericksen-Leslie system is concerned, many research works have been established on its well-posedness. In 2-D case, the existence of global weak solution for a simplified Ericksen-Leslie equation has been obtained by the authors in [20], where the domain is supposed to be bounded and smooth. The associated uniqueness problem was later studied by Lin-Wang in [21]. In [14] the author considered the same simplified Ericksen-Leslie equation but on the whole space $\mathbb{R}^2$. When the spatial domain is $\mathbb{R}^2$ and the model is not restricted to the simplified one studied in [20], the global existence of weak solutions for the Ericksen-Leslie system with general Oseen-Frank energy are also well studied (see [15, 16]). Amongst all the works in 2-D, global weak solutions have finite energy and are smooth except possibly at finitely many singularities. Compared with the 2-D case, our knowledge on the 3-D Ericksen-Leslie system is limited. In [28] the authors established the local well-posedness of the general Ericksen-Leslie system. For the sake of describing its maximal existence time interval, a blow-up criterion (same as the one in [17]) is given. With this criterion, the authors proceed to prove the global existence of the general Ericksen-Leslie system under the assumption that initial data is small in some Sobolev spaces. The spatial domain in [28] is $\mathbb{R}^3$. For the bounded smooth domain in $\mathbb{R}^3$, the authors in [23] also established a global existence result for weak solution of simplified Ericksen-Leslie equation. Different from [28], the consequence in [23] does not rely on the smallness of initial data in Sobolev spaces. Instead Lin-Wang made a geometrically small assumption in [23] for their initial data. More precisely by supposing that initial macroscopic orientation takes its image on the upper hemisphere, the simplified Ericksen-Leslie equation studied in [20] admits a global weak solution on any bounded smooth domain in $\mathbb{R}^3$, where initial data is only required to be in the natural energy space. For more detailed mathematical studies of nematic liquid crystals, readers are referred to [22].

Without macroscopic orientation, the Ericksen-Leslie system is reduced to the pure Navier-Stokes equation. It is well-known that the Navier-Stokes equation admits a vorticity-stream formulation (see [5]). For the 2-D viscous fluid, taking