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IMPLICIT DETERMINANT METHOD FOR SOLVING AN HERMITIAN EIGENVALUE OPTIMIZATION PROBLEM*

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Abstract

Implicit determinant method is an effective method for some linear eigenvalue optimization problems since it solves linear systems of equations rather than eigenpairs. In this paper, we generalize the implicit determinant method to solve an Hermitian eigenvalue optimization problem for smooth case and non-smooth case. We prove that the implicit determinant method converges locally and quadratically. Numerical experiments confirm our theoretical results and illustrate the efficiency of implicit determinant method.

Mathematics subject classification: 15A18, 65F15.
Key words: Eigenvalue optimization, Multiple eigenvalue, Non-smooth optimization, Implicit determinant method, Crawford number.

1. Introduction

Let $A(\omega) \in \mathbb{C}^{n\times n}$ be an Hermitian matrix which analytically depends on a parameter $\omega \in \mathbb{R}$. Let eigenvalues of $A(\omega)$ be sorted by $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_n(\omega)$. In this work, for a fixed integer l, $1 \leq l \leq n$, we restrict our attention to minimize or maximize $\varphi(\omega) = \lambda_l(\omega)$ in a bounded interval (a,b), that is

$$\min_{\omega\in(a,b)} \varphi(\omega) \quad \text{or} \quad \max_{\omega\in(a,b)} \varphi(\omega). \tag{1.1}$$

We assume that there exists a local extreme point ω^* in (a,b).

Eigenvalue optimization problem (1.1) has many applications. For examples, the computation of the stable radius [24] of a stable matrix, the computation of the H_∞ norm [13] of a linear system, the computation of the Crawford number [15] and quadratic constrained quadratic programming [25] for a frequently encountered case [8], can be converted into eigenvalue optimization problem (1.1).

Many methods solve the eigenvalue optimization problem which is more general than (1.1), say the parameter ω may be in high dimensional space. For example, Overton's method bases on successive quadratic programming [21]. The method of Mengi *et al.* bases on piecewise quadratic support functions [20]. Subspace method of Kangal *et al.* solves large-scale eigenvalue optimization problem [14]. All these methods require to solve an eigenvalue problem at each iteration step.

The implicit determinant method (IDM for short) was originally proposed by Spence and Poulton [23] for solving the nonlinear eigenvalue problem $H(\omega,\lambda)x = 0$, where $H(\omega,\lambda)$ is an

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Hermitian matrix function with respect to the parameter ω, and λ is the eigenvalue of this nonlinear eigenvalue problem. Later, Freitag and Spence [6] applied IDM to compute the stable radius of a stable matrix. The computation of stable radius can be transformed into eigenvalue optimization problem (1.1) which has a global minimizer ω^* existing in a bounded interval $(-2\|A\|, 2\|A\|)$, with $A(\omega) = \tilde{A} - \omega\tilde{C}$ and $l = N$, where $\tilde{A}, \tilde{C} \in \mathbb{C}^{2N\times 2N}$ are Hermitian matrices [24]. In [7], Freitag *et al.* applied IDM to compute the H_∞ norm. The computation of the H_∞ norm can also be converted into eigenvalue optimization problem (1.1) which has a global maximizer ω^* existing in a bounded interval, where $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_m(\omega)$ are finite generalized eigenvalues of the matrix pencil $(A(\omega), \mathrm{diag}(I, \mathbf{0}))$, $A(\omega) = \tilde{A} - \omega\tilde{C}$, $l = 1$, $\tilde{A}, \tilde{C}$ are Hermitian matrices [7]. Compared with most methods for eigenvalue optimization problem, the computation cost of IDM is to solve linear systems rather than eigenvalue problem at each iteration step. However, in [6, 7], the conditions for IDM include 1. $A(\omega)$ is a linear matrix function, that is $A(\omega) = \tilde{A} - \omega\tilde{C}$, 2. $\varphi(\omega^*)$ is a simple eigenvalue of $A(\omega^*)$.

In this paper, we first generalize IDM to solve eigenvalue optimization problem (1.1) where $A(\omega)$ is a nonlinear Hermitian matrix function of ω. The generalization is almost straightforward, and the purpose is to introduce the IDM. However, the sequence generated by IDM only converges to $(\omega^*, \varphi(\omega^*))$ if $\varphi(\omega^*)$ is a simple eigenvalue of $A(\omega^*)$, which implies $\varphi(\omega)$ is smooth at ω^*. Secondly, we generalize the IDM for the case that $\varphi(\omega^*)$ is an eigenvalue of $A(\omega^*)$ with multiplicity 2, and in this case, $\varphi(\omega)$ is usually non-smooth at ω^*. We prove that this generalized IDM converges locally quadratically. Similar to previous IDM, our generalized IDM only needs to solve linear systems at each iteration step, and in turn, IDM is more effective than other methods such as subspace method.

This paper is organized as follows. In Section 2, we apply IDM to solve eigenvalue optimization problem (1.1) for the case that $\lambda^* = \varphi(\omega^*)$ is a simple eigenvalue of $A(\omega^*)$, where $A(\omega)$ is a nonlinear matrix function of ω. Under the condition that $\lambda^* = \varphi(\omega^*)$ is an eigenvalue of $A(\omega^*)$ with multiplicity 2, we generalize IDM to solve (1.1) and prove that it converges locally quadratically in Section 3. In Section 4, numerical experiments confirm the rate of convergence established in theory and show the efficiency of generalized IDM.

2. Implicit Determinant Method for Smooth Case

We first introduce the relation between smoothness of $\varphi(\omega)$ and multiplicity of eigenvalues of $A(\omega)$ (see e.g., [9, 16, 18, 22]).

Theorem 2.1 ([9, Theorem S6.3]). *Let $A(\omega) \in \mathbb{C}^{n\times n}$ be an Hermitian matrix-valued function that depends on $\omega \in \mathbb{R}$ analytically. Then there exist scalar functions $\tilde{\lambda}_1(\omega), \ldots, \tilde{\lambda}_n(\omega)$ and a matrix-valued function $V(\omega) = [v_1(\omega), \ldots, v_n(\omega)]$, which are analytic for ω and possess the following properties for every $\omega \in \mathbb{R}$:*

$$A(\omega) = V(\omega)\,\mathrm{diag}(\tilde{\lambda}_1(\omega), \ldots, \tilde{\lambda}_n(\omega))V(\omega)^{\mathrm{H}}, \quad V(\omega)^{\mathrm{H}}V(\omega) = I.$$

The left subplot of Fig. 2.1 depicts these analytic eigenvalue curves $\tilde{\lambda}_i(\omega)$, $i = 1, \ldots, n$. Now we sort these eigenvalue curves as $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_n(\omega)$. The right subplot of Fig. 2.1 depicts these sorted eigenvalue curves $\lambda_i(\omega)$. From Theorem 2.1, we can see that, $\lambda_i(\omega)$, $i = 1, \ldots, n$, are continuous and piecewise analytic [20]. For a fixed point ω, if $\lambda_i(\omega)$ is a simple eigenvalue of $A(\omega)$, then $\lambda_i(\omega)$ is analytic at ω. If $\lambda_i(\omega)$ is not differentiable at ω, then $\lambda_i(\omega)$ must be a multiple eigenvalue of $A(\omega)$.