Journal of Computational Mathematics
Vol.41, No.6, 2023, 1246–1280.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2203-m2021-0031

A MODIFIED WEAK GALERKIN FINITE ELEMENT METHOD FOR SINGULARLY PERTURBED PARABOLIC CONVECTION-DIFFUSION-REACTION PROBLEMS*

Suayip Toprakseven[1)]
Faculty of Engineering, Department of Computer Science, Artvin Çoruh University, Artvin, 08100, Turkey
Email: topraksp@artvin.edu.tr
Fuzheng Gao
School of Mathematics, Shandong University, Jinan 250100, China
Email: fzgao@sdu.edu.cn

Abstract

In this work, a modified weak Galerkin finite element method is proposed for solving second order linear parabolic singularly perturbed convection-diffusion equations. The key feature of the proposed method is to replace the classical gradient and divergence operators by the modified weak gradient and modified divergence operators, respectively. We apply the backward finite difference method in time and the modified weak Galerkin finite element method in space on uniform mesh. The stability analyses are presented for both semi-discrete and fully-discrete modified weak Galerkin finite element methods. Optimal order of convergences are obtained in suitable norms. We have achieved the same accuracy with the weak Galerkin method while the degrees of freedom are reduced in our method. Various numerical examples are presented to support the theoretical results. It is theoretically and numerically shown that the method is quite stable.

Mathematics subject classification: 65N15, 65N30, 35J50.
Key words: The modified weak Galerkin finite element method, Backward Euler method, Parabolic convection-diffusion problems, Error estimates.

1. Introduction

In this paper, we propose a modified weak Galerkin finite element method (MWG-FEM) for the following parabolic convection-diffusion problem:

$$\begin{aligned} &\partial_t u - \varepsilon\Delta u + \nabla\cdot(\mathbf{b}u) + cu = f && \text{in} \quad Q_T = \Omega\times(0,T],\\ &u = 0 && \text{on} \quad \partial\Omega\times(0,T],\\ &u = u_0 && \text{in} \quad \Omega\times\{0\}, \end{aligned} \tag{1.1}$$

where $\varepsilon \in (0,1]$ is a small parameter and Ω is a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$, $\partial_t u = \frac{\partial u}{\partial t}$ and $u_0 \in L^2(\Omega)$. For the well-posedness of the problem [24], we assume that $\mathbf{b}, c$ and f are smooth functions, $\mathbf{b} \in [W^{1,\infty}(\Omega)]^2$ and for some constant a_0 such that

$$c + \frac{1}{2}\nabla\cdot\mathbf{b} \geq a_0 > 0. \tag{1.2}$$

* Received January 28, 2021 / Revised version received June 11, 2021 / Accepted March 11, 2022 / Published online December 7, 2022 /
1) Corresponding author

Convection-diffusion equations are commonly used to describe a wide range of differential equations arising from the mathematical modeling of real word problems in science and engineering involving fluid, petroleum simulation, groundwater contamination and gas dynamics [2,3,30], etc. Applications generally involve time-dependent convection-dominated problems for the mathematical modeling of physical processes. It is well known that the solution of the singularly perturbed convection-diffusion problems possess boundary or interior layers. It is well known that these layers lead to unsatisfactory numerical solutions with non-physical oscillations when the conventional numerical methods such as finite difference (FD) methods and the standard finite element methods are applied. To recover these non-physical oscillations, some stabilization methods have been proposed over the last decades, including streamline-upwind Petrov-Galerkin (SUPG) methods proposed by Hughes and Brooks [4], local projection stabilization method [15, 19] and the interior penalty method [35]. However, there are some disadvantages of these methods for convection-dominated problems. For instance, the popular SUPG methods have the stabilization term which includes many terms for the time dependent problems. Moreover, they produce overshoots and undershoots near the layer region. A large of papers has been devoted to the numerical methods for convection-diffusion problems on some layer adapted meshes in the literature [18]. Unfortunately, the location of the layer must be known in prior in order to use the layer adapted meshes. The layers may move as time varies in the parabolic convection-diffusion problems. This leads to use fitted operator methods for the numerical solutions of unsteady convection-diffusion-reaction equations.

Wang and Ye [32] first introduced the weak Galerkin finite element method (WG-FEM) and analyzed for numerical solution of second order differential equations. The WG-FEMs introduce a space of weak functions, weak gradient and weak divergence on the space of completely discontinuous piecewise polynomials. The weak functions in WG-FEMs consist of the form $u = \{u_0, u_b\}$ with $u = u_0$ inside of the element and $u = u_b$ on the boundary of the element. Later on, WG finite element methods have further been presented for a large variety of PDEs including the implementation results [21], parabolic problems [16], the Helmholtz equations with high wave numbers in [22] and the time fractional reaction-diffusion-convection problems in [27]. The weak gradient and weak divergence operators have been introduced for convection-dominated problems in [5] and [17]. The WG-FEM has been studied and analyzed for time-dependent convection-diffusion equations with convection term in non-conservation form based on these newly defined operators [34]. While the formulation of WG-FEM is simple and parameter-free, it adds more degrees of freedom since it has two components for each function in the approximation space. In order to reduce the degrees of freedom in the formulation of the WG-FEM, a modified WG-FEM (MWG-FEM) introduced in [31] eliminates u_b from the space of weak functions and uses the average $\{u_0\}$ of the u_0 on the boundary of element. As a result, the weak functions in the MWG-FEM of the form $u = \{u_0, \{u_0\}\}$ and for simplicity we denote by u. As the WG-FEM, the MWG-FEM is a parameter free method and it has the same degrees of freedom as the discontinuous Galerkin (DG) methods. In other words, the MWG-FEM inherits from the properties of the WG-FEM with the reduced number of unknowns in the associated discrete systems. Compared to DG methods, the formulation of the MWG-FEM is simple, symmetric and the resulting system is positive definite while they have the same same finite element space and there is no need a large penalization parameter for the MWG-FEM. MWG-FEMs have been further developed for a variety of PDEs such as convection-diffusion problems [12], parabolic equations [11], Stokes equations [20, 26], convection-diffusion problems in one dimension [28] and in higher dimension with weakly imposed boundary condition [9].