Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2022-0272

Vol. **34**, No. 4, pp. 934-954
October 2023

A Variational Neural Network Approach for Glacier Modelling with Nonlinear Rheology

Tiangang Cui[1], Zhongjian Wang[2,*] and Zhiwen Zhang[3]

[1] *School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia.*
[2] *Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371.*
[3] *Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong SAR, China.*

Received 26 October 2022; Accepted (in revised version) 6 February 2023

Abstract. We propose a mesh-free method to solve the full Stokes equation for modeling the glacier dynamics with nonlinear rheology. Inspired by the Deep-Ritz method proposed in [13], we first formulate the solution to the non-Newtonian Stokes equation as the minimizer of a variational problem with boundary constraints. Then, we approximate its solution space by a deep neural network. The loss function for training the neural network is a relaxed version of the variational form, in which penalty terms are used to present soft constraints due to mixed boundary conditions. Instead of introducing mesh grids or basis functions to evaluate the loss function, our method only requires uniform sampling from the physical domain and boundaries. Furthermore, we introduce a re-normalization technique in the neural network to address the significant variation in the scaling of real-world problems. Finally, we illustrate the performance of our method by several numerical experiments, including a 2D model with the analytical solution, the Arolla glacier model with realistic scaling and a 3D model with periodic boundary conditions. Numerical results show that our proposed method is efficient in solving the non-Newtonian mechanics arising from glacier modeling with nonlinear rheology.

AMS subject classifications: 35A15, 65J15, 68T99, 70K25, 76A05

Key words: Deep learning method, variational problems, mesh-free method, non-Newtonian mechanics, nonlinear rheology, glacier modelling.

1 Introduction

In recent years, deep neural networks (DNNs) have achieved unprecedented levels of success in a broad range of areas such as computer vision, speech recognition, natural

*Corresponding author. *Email addresses:* `tiangang.cui@sydney.edu.au` (T. Cui), `zhongjian.wang@ntu.edu.sg` (Z. Wang), `zhangzw@hku.hk` (Z. Zhang)

 ©2023 Global-Science Press

language processing, and health sciences, producing results comparable or superior to human experts [17, 31]. The impacts have reached physical sciences where traditional first-principle based modeling and computational methodologies have been the norm. Thanks in part to the user-friendly open-source computing platforms from industry, e.g. TensorFlow and PyTorch, there have been vibrant activities in applying deep learning tools for scientific computing, such as approximating multivariate functions, solving ordinary/partial differential equations (ODEs/PDEs) and inverse problems using DNNs; see, e.g. [2,13,19,28,51,60,65] and references therein. There are many classical works on the approximation power of neural networks; see e.g. [11, 14, 22, 47]. For recent works on the expressive (approximation) power of DNNs; see, e.g. [10,36,41,54,55,63]. In [19], the authors showed that DNNs with rectified linear unit (ReLU) activation function and enough width/depth contain the continuous piece-wise linear finite element space. Thus, one can represent a solution of PDE using the ReLU-DNN.

Solving ODEs or PDEs by a neural network (NN) approximation is known in the literature dating back at least to the 1990's; see e.g. [30,32,40]. The main idea in these works is to train NNs to approximate the solution by minimizing the residual of the ODEs or PDEs, along with the associated initial and boundary conditions. These early works estimate neural network solutions on a fixed mesh. Recently DNN methods are developed for Poisson and eigenvalue problems with a variational principle characterization (deep Ritz, [13]), for a class of high-dimensional parabolic PDEs with stochastic representations [18], for advancing finite element methods [7, 8, 20], for nonconvex energy minimization in simulating martensitic phase transitions [9], and for learning and generating invariant measures of stochastic dynamical systems with parameters [59]. The physics-informed neural network (PINN) method [49] and a deep Galerkin method (DGM) [56] compute PDE solutions based on their physical properties. For parametric PDEs, a deep operator network (DeepONet) learns operators accurately and efficiently from a relatively small dataset based on the universal approximation theorem of operators [37]; a Fourier neural operator method [33] directly learns the mapping from functional parametric dependence to the solutions of a family of PDEs. In [1, 64], weak adversarial network methods are studied for weak solutions and inverse problems, see also related studies on PDE recovery from data via DNN [34, 35, 48, 61] among others. In the context of surrogate modeling and uncertainty quantification (UQ), DNN methods include Bayesian deep convolutional encoder-decoder networks [65], deep multi-scale model learning [58], physics-constrained deep learning method [66], see also [27,28,54,62] and references therein.

In this work, we present a deep learning method for solving problems in non-Newtonian mechanics that obey certain variational principles. In particular, we focus on nonlinear Stokes problems in which the viscosity nonlinearly depends on the strain rate. This type of problems plays a fundamental role in modelling geodynamic processes, for instance, the dynamics of glaciers [24, 46] and mantle convection [39, 52]. The solutions of these problems typically face a combination of challenges, such as the presence