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Flux Globalization Based Well-Balanced Path-Conservative Central-Upwind Scheme for the Thermal Rotating Shallow Water Equations

Yangyang Cao[1], Alexander Kurganov[2] and Yongle Liu[3,4,*]

[1] *Faculty of Computational Mathematics and Cybernetics, Shenzhen MSU-BIT University, Shenzhen, 518172, China.*
[2] *Department of Mathematics, Shenzhen International Center for Mathematics and Guangdong Provincial Key Laboratory of Computational Science and Material Design, Southern University of Science and Technology, Shenzhen, 518055, China.*
[3] *Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland.*
[4] *Department of Mathematics, Southern University of Science and Technology, Shenzhen, 518055, China.*

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Abstract. We present an extension of the flux globalization based well-balanced path-conservative central-upwind scheme to the one- and two-dimensional thermal rotating shallow water equations. The scheme is well-balanced in the sense that it can exactly preserve a variety of physically relevant steady states. In the one-dimensional case, it can preserve different "lake-at-rest" equilibria, thermo-geostrophic equilibria, as well as general moving-water steady states. In the two-dimensional case, preserving general moving-water steady states is difficult, and to the best of our knowledge, none of existing schemes can achieve this ultimate goal. The proposed scheme can exactly preserve the x- and y-directional jets in the rotational frame as well as certain genuinely two-dimensional equilibria. Furthermore, our approach employs a path-conservative technique for discretizing nonconservative product terms, which are incorporated into the global fluxes. This allows the developed scheme to exactly preserve some of the discontinuous steady states as well. We provide a number of numerical examples to demonstrate the advantages of the proposed scheme over some alternative finite-volume methods.

AMS subject classifications: 76M12, 65M08, 86-08, 86-10, 86A10, 35L67

Key words: Thermal rotating shallow water equations, well-balanced schemes, flux globalization, path-conservative central-upwind schemes.

*Corresponding author. *Email addresses:* caoyangyang@smbu.edu.cn (Y. Cao), alexander@sustech.edu.cn (A. Kurganov), liuyl2017@mail.sustech.edu.cn (Y. Liu)

1 Introduction

In this paper, we consider the thermal rotating shallow water (TRSW) equations, which in the two-dimensional (2-D) case read as (see, e.g., [50]):

$$\begin{cases} h_t+(hu)_x+(hv)_y=0, \\ q_t+\Big(hu^2+\dfrac{b}{2}h^2\Big)_x+(huv)_y=fhv-hbZ_x, \\ p_t+(huv)_x+\Big(hv^2+\dfrac{b}{2}h^2\Big)_y=-fhu-hbZ_y, \\ (hb)_t+(hub)_x+(hvb)_y=0, \end{cases} \tag{1.1}$$

where x and y are zonal and meridional coordinates, t is the time, $h(x,y,t)$ denotes the thickness of the fluid layer, $u(x,y,t)$ and $v(x,y,t)$ are the zonal and meridional velocities, $q:=hu$ and $p:=hv$ are the corresponding discharges, $Z(x,y)$ stands for bottom topography, and $b(x,y,t)$ is the buoyancy variable. In the oceanic context, $b=g\rho/\rho_0$, where g is the acceleration due to gravity, ρ and ρ_0 are the variable and constant parts of the water density, respectively. In the atmospheric context, the densities ρ and ρ_0 should be replaced with the potential temperatures θ and θ_0. Finally, $f(y)=f_0+\beta y$ is the Coriolis parameter, where f_0 and β are positive constants.

The TRSW model (1.1) was first introduced in [77] and since then, it has been rediscovered and extensively used in the literature related to both atmospheric and oceanographic studies. Notably, this model has been successfully applied to investigate the boundary layer in the atmosphere [54, 71], as well as the mixed layer in the ocean [60, 70, 81]. In [49], we have used a more involved modification of the system (1.1), the so-called moist-convective TRSW equations (derived in [7, 47]) to model the interaction of tropical cyclone-like vortices with sea-surface temperature anomalies and topography. In addition, recent applications of the TRSW model include its use in planetary atmospheres [28, 78], and it has once again been rediscovered in the context of testing general circulation models [83]. The widespread adoption and multiple applications of the TRSW model highlight its versatility and relevance in understanding various geophysical and planetary phenomena.

It is commonly understood that steady-state solutions of (1.1) are of great particular importance as large-scale dynamical features in the atmosphere and ocean are usually closed to equilibria, and many practically relevant waves can be viewed as small perturbations of those equilibria. Therefore, a good numerical method should be able to accurately capture these steady-state solutions. In particular, one is interested in developing well-balanced (WB) numerical schemes, which are capable of exactly preserving (some of) the steady states. In addition, a good numerical method should be able to resolve sharp temperature and pressure fronts, since a description of such fronts was a part of the initial motivation to introduce the TRSW model; see [31,82]. Besides this, as the terms $-hbZ_x$ and $-hbZ_y$ appearing on the right-hand side (RHS) of (1.1) are nonconservative, a