JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **36**, No. 4, pp. 331-348

doi: 10.4208/jpde.v36.n4.1
December 2023

Lipschitz Continuity and Explicit Form of Solution in a Class of Free Boundary Problem with Neumann Boundary Condition

SAADI Abderachid*

Department of Mathematics, University of Msila, Algeria.
Laboratory of Nonlinear Partial Differential Equations and History of Mathematics, ENS, Kouba, Algeria.

Received 3 June 2020; Accepted 13 May 2023

Abstract. We consider a class of free boundary problems with Neumann boundary conditions. We would like to give certain results with regularity of solutions (mainly the local interior and boundary Lipschitz continuity). We will also show an explicit form of solution under well-specified conditions.

AMS Subject Classifications: 35J15, 35R35

Chinese Library Classifications: O175.29

Key Words: Lipschitz continuity; free boundary; Neumann boundary condition.

---

1 Introduction

In this article, we are interested in studying a free boundary problem with Neumann boundary conditions, whose weak formulation is as follows

$$\begin{cases} \text{Find } (u,\chi)\in H^1(\Omega)\times L^\infty(\Omega) \text{ such that:} \\ \textbf{(i)}\quad u\geq 0, 0\leq \chi\leq 1, u(1-\chi)=0 \quad \text{a.e. in } \Omega, \\ \textbf{(ii)}\quad u=0 \quad \text{on } \Gamma_2, \\ \textbf{(iii)}\ \displaystyle\int_\Omega (a(x)\nabla u+\chi H(x))\nabla\xi \leq \int_{\Gamma_3}\beta(x,\varphi-u)\xi \mathrm{d}\sigma(x), \quad \forall \xi\in H^1(\Omega),\ \xi\geq 0 \text{ on } \Gamma_2, \end{cases}$$

*Corresponding author. *Email addresses:* `abderrachid.saadi@univ-msila.dz`, or `rachidsaadi81@gmail.com`. (A. Saadi)

where Ω be a C^1 bounded domain of $\mathbb{R}^n$, and $\Gamma_2,\Gamma_3=(\Gamma_{3,i})_{1\le i\le N}$ are relatively open connected subsets of $\partial\Omega$ and $\Gamma_1=\partial\Omega\backslash(\Gamma_2\cup\Gamma_3)$, $x=(x',x_n)$, $e=(0,\dots,0,1)$, $a(x)=(a_{ij}(x))$ is an n by n matrix satisfying these conditions below for $\Lambda,\lambda>0$

$$\begin{aligned}
&a(x)z\cdot z\ge\lambda|z|^2, && \text{for all } z\in\mathbb{R}^n,\\
&|a_{ij}(x)|\le\Lambda, && \text{for all } x\in(\Omega\cup\Gamma_3),\ \text{for all } i,j=1,\cdots,n,\\
&\left|\frac{\partial a_{i,j}}{\partial x_i}(x)\right|\le kn\sqrt{n}, && \text{for all } \ x\in(\Omega\cup\Gamma_3),\ \text{for all } i,j=1,\cdots,n.
\end{aligned}\tag{1.1}$$

In addition, $H(x)$ is a $C^1(\overline{\Omega})$ vector function not necessarily depending on the matrix $a(x)$, satisfying these conditions follow for the positive constants $\underline{H},\overline{H}$ and c

$$\begin{aligned}
&0\le\operatorname{div}H\le c, && \text{a.e. } x\in\Omega,\\
&|H_i(X)|\le\overline{h}, && i=1,\cdots,n-1,\ \text{a.e. } x\in\Omega,\\
&0<\underline{H}\le H_n(X)\le\overline{H} && \text{a.e. } x\in\Omega,\\
&H(x)\cdot\nu\neq0 && \text{for all } \ x\in\partial\Omega.
\end{aligned}\tag{1.2}$$

For a.e. $x\in\Gamma_3$, the function $\beta(x,\cdot)$ is a nonnegative, nondecreasing function satisfying

$$\begin{aligned}
&\text{For all } v\in\mathbb{R}, x\mapsto\beta(x,v) \text{ is } d\sigma-\text{measurable},\\
&\text{there exists } b>0 \text{ such that; } \beta(x,\cdot)\le b \text{ a.e. } x\in\Gamma_3,\\
&\text{for a.e. } x\in\Gamma_3,\ \beta(\cdot,v) \text{ is Lipschitz continuous function.}
\end{aligned}\tag{1.3}$$

And φ is a nonnegative Lipschitz continuous function.

This problem describes of some free boundaries, such as the aluminum electrolysis problem [1], the problem of lubrication [2], and the dam problem with leaky boundary conditions [3–6]. The separation in the dam problem is between the part of the domain where the water is and the rest. The separation in the case of the aluminum electrolysis problem is between the liquid and solid region. The separation in the lubrication problem is between the nil-pressure zone and the other region.

In [7], Carrillo and Chipot showed the existence of solution, the lower semi-continuity of the function representing the free boundary and the uniqueness of the solution, which they called "S3-connected solution" for the Dirichlet condition on $\Gamma_1\cup\Gamma_2$. The existence proved in [3] in the case $a(x)=I_n$ and $H(x)=a(x)e$.

In [8], Chipot considered the problem in the case where $H(x)=h(x)\cdot e$ with $e=(1,0)$ and $h_{x_1}\ge0$ in $D'(\Omega)$. He proved that the free boundary is a graph of a continuous function $x_1=\phi(x_2)$. Challal and Lyaghfouri [9] proved the same result under weaker assumptions. This result is generalized in the article published by Saadi [10].

In [11], Lyaghfouri generalized the result shown by Carrillo and Chipot in [7] to the case

$$a(x)=\begin{pmatrix} a_{11}(x) & 0\\ a_{21}(x) & a_{22}(x)\end{pmatrix},\quad H(x)=a(x)e,\quad\text{and}\quad \frac{\partial a_{22}}{\partial x_2}\ge0\quad\text{in } \mathcal{D}'(\Omega).$$