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Blowup of the Solutions for a Reaction-Advection-Diffusion Equation with Free Boundaries

YANG Jian*

School of Mathematical Sciences, University of Jinan, Jinan 250022, China.

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Abstract. We investigate a blowup problem of a reaction-advection-diffusion equation with double free boundaries and aim to use the dynamics of such a problem to describe the heat transfer and temperature change of a chemical reaction in advective environment with the free boundary representing the spreading front of the heat. We study the influence of the advection on the blowup properties of the solutions and conclude that large advection is not favorable for blowup. Moreover, we give the decay estimates of solutions and the two free boundaries converge to a finite limit for small initial data.

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1 Introduction

We study the blowup solution for the following reaction-advection-diffusion equation

$$\begin{cases} u_t = u_{xx} - \beta u_x + u^p, & g(t) < x < h(t),\ t > 0, \\ u(t,g(t)) = 0,\ g'(t) = -\mu u_x(t,g(t)), & t > 0, \\ u(t,h(t)) = 0,\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ -g(0) = h(0) = h_0,\ u(0,x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{1.1}$$

where $x = g(t), h(t)$ are free boundaries to be determined. $p > 1$ and h_0, μ, β are some given positive constants. The initial function u_0 satisfies

$$u_0 \in C^2([-h_0, h_0]), \quad u_0 > 0 \text{ in } (-h_0, h_0) \quad \text{and} \quad u_0(-h_0) = u_0(h_0) = 0. \tag{1.2}$$

*Corresponding author. *Email addresses:* yangjian86419@126.com (J. Yang)

Problem (1.1) may be viewed as describing the heat transfer and temperature change in chemical reaction, $u(t,x)$ may represent the temperature over a one-dimensional region and the initial function $u_0(x)$ stands for the temperature distribution in each point at the initial time which occupy an initial region $(-h_0,h_0)$.

When $\beta=0$ (i.e. there is no advection in the environment), the qualitative properties of the problem (1.1) were studied by Souplet and his collaborators [1–4], they study the blowup phenomenon of reaction-diffusion equation with a free boundary and prove the solution blows up in finite time in L^∞ norm and all global solutions are bounded and decay uniformly to 0.

Our concern is to discuss the blowup properties of the solutions of (1.1) with $\beta>0$, which means the heat transfer is affected by advection. In the process of chemical reactions, we may use the free boundary to represent the spreading front of the heat. We know that chemical reactions with a high initial temperature produce a lot of heat, which will increase the temperature of chemical reactions, thus accelerating the rate of chemical reactions, and as a result making the reaction temperature higher and higher and even converge to infinity in finite time. Mathematically, it means the solutions will blow up. In the field of physics [5], the advective heat transfer is an important physical process. Advective heat transfer is caused by relative displacement of particles during fluid movement. In some cases, the fluid motion that occurs under the influence of external forces is called forced motion of fluid. Heat transfer and fluid motion are inextricably linked because of advective heat transfer. Therefore, advection due to external forces is a very important factor affecting heat transfer.

The main purpose of this paper is to investigate the effect of the advection on the properties of blowup solution of (1.1) provided the initial datum has compact supports. Our results indicate that larger advection makes blowup more difficult. We denote T^* as the blowup time throughout this paper:

$$\begin{aligned} T^*=T^*(u_0):=\sup\{T>0:\ &\text{the classical solutions are bounded in } [0,T]\\ &\text{for the initial data } u_0\}. \end{aligned}$$

If $T^*<\infty$, one has

$$\limsup_{t\to T^*}\|u(t,\cdot)\|_{L^\infty([g(t),h(t)])}=\infty.$$

The plan of the paper is as follows: some preliminary results, including the existence and the comparison principle of (1.1) are gathered in Section 2. Blowup and vanishing results are proved in Section 3.

2 Preliminaries

We begin by recalling the local existence and uniqueness result which can be proved by the similar method as in [6–8].